A Short Note on Synchronisation in Open Systems

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Abstract

This note considers the central activities of logistic network nodes which are typical examples of open systems exhibiting synchronisation aspects. We model a node's behaviour by means of a simple GSPN and show that a steady state distribution does not exist, irrespective of the parameters for the arrival streams. This effect shows an intrinsic problem of logistic networks and generally of open systems with synchronisation. We show additionally that imposing bounds on the population size is not helpful for analysis objectives like capacity planning.

1 Introduction

Typical nodes of logistic networks are governed by transhipping processes. Consider one such node. Goods are delivered from outside, temporarily stored, and eventually collected and shipped out. Capacity planning is a standard task when designing such systems, and it appears a natural approach to start model-based analysis assuming an open and unlimited system. Hopefully results like mean buffer sizes, their variances etc. will give the analyst first hints at designing the logistic node.

As an example let us consider a large goods distribution center (GDC) where vehicles arrive, governed by Poisson arrival streams, either attempting to unload goods to an internal storage or load goods from that storage. For simplicity we assume that exactly one storage unit is delivered respectively loaded and that the time for the transhipping activities can be neglected. Vehicles have to wait when their demands cannot be satisfied because of an empty or full storage, the size and content of the storage being measured in storage units.

2 An open generalized stochastic Petri net

Let the arrival rate of those vehicles delivering a unit be λ and the arrival rate of those loading a unit be μ. For simplification we do not need to distinguish between the transportation means which are going to deliver a unit and the stock's content, since the time for handling goods was assumed to be negligible (immediate in the model). A model of our GDC is the generalized stochastic Petri net (GSPN; cf. e.g. /BK96/) of Figure 2-1.

* This research was supported by the Deutsche Forschungsgemeinschaft as part of the Collaborative Research Center „Modelling of large logistic networks“ (559).

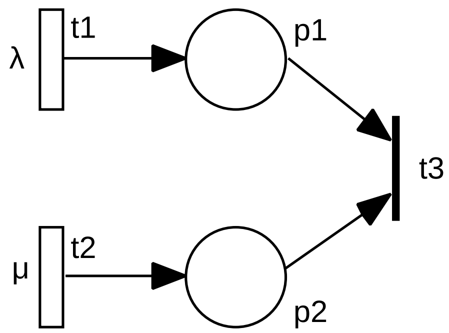

Figure 2-1 An open GSPN

Tokens at place p1 represent the storage content. Tokens at p2 represent those vehicles attempting to load goods. The immediate transition t3 implies that whenever $M(p1) > 0$, we have $M(p2) = 0$ and similarly $M(p2) > 0$ implies $M(p1) = 0$. Thus $M(p1) - M(p2)$ is a sufficient state descriptor and the potentially two dimensional state space of the GSPN can be represented in one dimension as shown in Figure 2-2.

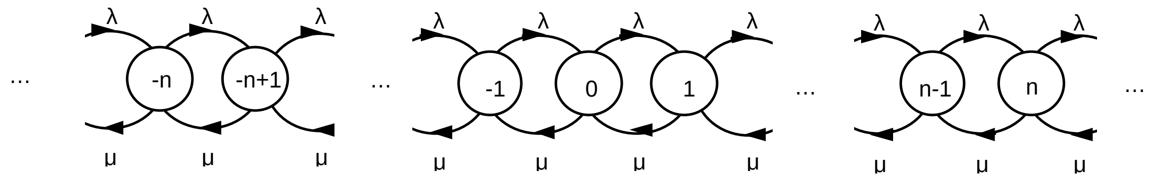

Figure 2-2 Markov chain of the GSPN

The global balance equations are given by

(1a) $\pi_i(\lambda + \mu) = \lambda\pi_{i-1} + \mu\pi_{i+1}$ $\quad i = -\infty, \ldots, +\infty$

which rewrites as

(1b) $\lambda(\pi_i - \pi_{i-1}) = \mu(\pi_{i+1} - \pi_i)$ $\quad i = -\infty, \ldots, +\infty$

If system (1) could be solved subject to $0 \le \pi_i \le 1$, for all i, and $\sum \pi_i = 1$, then a stationary distribution would in fact exist and would be given by this solution. However, it is not difficult to verify that such a solution cannot exist. Define $\tau_i = \pi_{i+1} - \pi_i$. With that (1) rewrites to $\tau_i = \frac{\lambda}{\mu}\tau_{i-1}$ giving $\tau_i = \left(\frac{\lambda}{\mu}\right)^i \tau_0$ implying $\pi_{i+1} - \pi_i = \left(\frac{\lambda}{\mu}\right)^i (\pi_1 - \pi_0)$ for $i = -\infty, \ldots, +\infty$.

We can now distiguish the following cases:

1) $\pi_1 = \pi_0$ implying $\pi_{i+1} = \pi_i, \ \forall i$ showing that $\sum_{i=-\infty}^{\infty} \pi_i = 1$ can not be satisfied.

2) $\pi_1 \ne \pi_0$. Consider $\pi_{n+1} - \pi_{-n} = \sum_{i=-n}^{n} (\pi_{i+1} - \pi_i) = \sum_{i=-n}^{n} \left(\frac{\lambda}{\mu}\right)^i (\pi_1 - \pi_0)$ for an arbitrary value of n.

   If $\pi_1 > \pi_0$ we can conclude $\pi_{n+1} > n(\pi_1 - \pi_0)$ contradicting $\pi_{n+1} < 1$ and assuming $\pi_1 < \pi_0$ implies $\pi_{n+1} < n(\pi_1 - \pi_0) + 1$ leading to a further contradiction for arbitrarily large n, namely $\pi_{n+1} < 0$.

The reader should note that all conclusions hold (and thus no steady state distribution exists) irrespective of the values of λ and μ!

We may now argue that the assumption of infinite buffers for vehicles and goods is not realistic and that choosing some (possibly huge) upper bound would solve our analysis problems.
So, for simplicity, let us assume that the capacities of both places p1 and p2 are restricted by some value n. Then a steady state distribution of our (now finite) one dimensional Markov chain of course exists, and is given by

$$\pi_i = \rho^i (1-\rho) \frac{\rho^n}{\left(1-\rho^{(2n+1)}\right)} \text{ with } \rho = \frac{\lambda}{\mu}$$

(cf. an M/M/1/N-system /GK84/), with an average value of $N = M(p1) - M(p2)$ of

$$E_n[N] := \sum_{i=-n}^{n} i \times \pi_i = \frac{n\rho^{2n+2} - (n+1)\rho^{2n+1} + (n+1)\rho - n}{(\rho - 1)\left(\rho^{(2n+1)} - 1\right)}.$$

Surely $E_n[N] = 0$, for all n, if $\rho = 1$, since $\pi_i = \frac{1}{2n+1}, \quad i$. The variance of N in this case is $VAR[N] = \frac{1}{3} n(n+1)$ showing that any attempt for dimensioning our formerly open system depends on the bound which has been introduced merely as a modelling artefact, in order to get an analysable model!

Furthermore it is also interesting to have a look at $E_n[N]$ for $\rho \neq 1$. Since all capacities are finite, we can now also consider cases where λ and μ differ, however slightly. E.g., if we approve of situations in which it is more likely to find goods in the stock (which makes sense when considering loading vehicles as customer demands), we would reflect upon arrival streams with $\lambda > \mu$. E.g., let us have a closer look at ρ=1.01. Figure 2-3 shows the dependence of $E_n[N]$ on n, again illustrating that the analysis results totally depend on the arbitrary choice of the upper bound. The reader should note that for this case $E_n[N]$ tends to n as n increases; in general, we have

$$\lim_{n\to\infty} \frac{E_n[N]}{n} = \begin{cases} -1 & 0 < \rho < 1 \\ 0 & \rho = 1 \\ 1 & \rho > 1 \end{cases}.$$

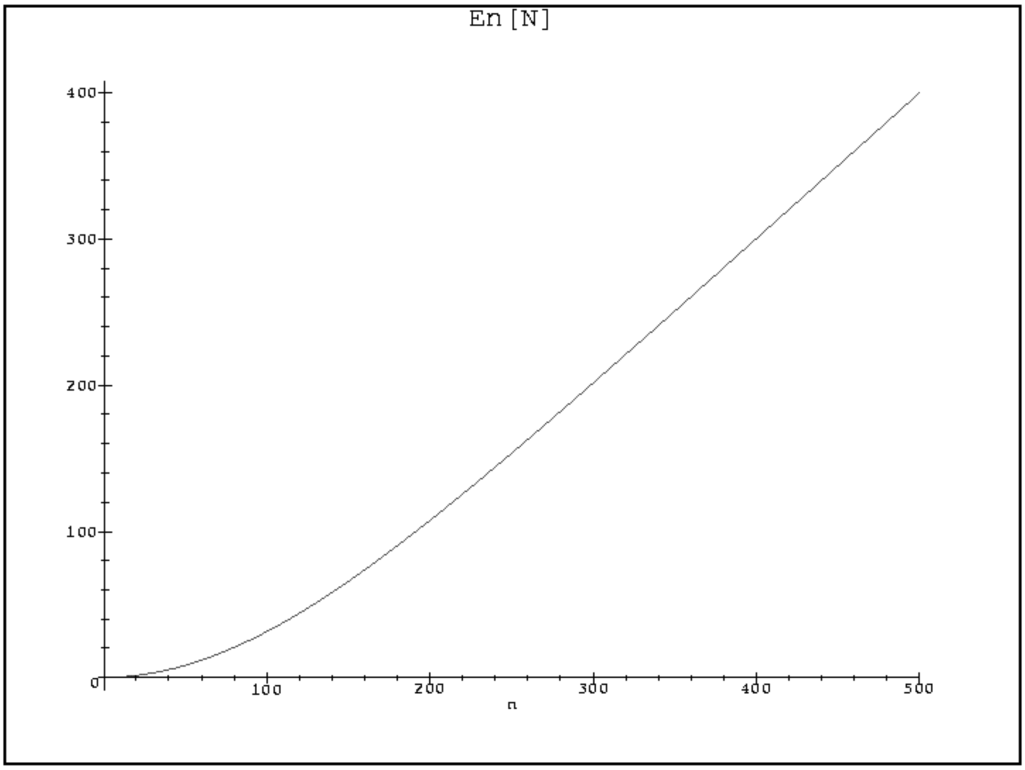

Figure 2-3 Mean value for ρ=1.01 and different values of n

3 Conclusions

The net of Figure 2-1 shows the core activities of an open system with synchronisation. As we have seen the possible non-existence of the steady state distribution is an intrinsic problem in such systems. Even assuming upper bounds, thus ensuring steady state, does not make sense for supporting analysis objectives like capacity planning.

4 References

/BK96/ F. Bause, P.S. Kritzinger: Stochastic Petri Nets – An Introduction to the Theory, Vieweg Verlag, 1996.

/GK84/ B.W. Gnedenko, D. König: Handbuch der Bedienungstheorie II, Akademie-Verlag, Berlin, 1984.