An Improved Method for Bounding Stationary Measures of Finite Markov Processes

Peter Buchholz
Informatik IV, University of Dortmund

- Basic Assumptions
- Bounding Approach by Courtois/Semal
- Improved Bounds
- A new Bounding Algorithm
- Conclusions and Outlook

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Basic Definitions

We consider Markov Chains in discrete time with

- State space $S = \{1, \ldots, n\}$
- Reward vector $r \in \mathbb{R}_+^n$
- A lower bound on the transition probabilities $L \in \mathbb{R}^{n \times n}$ with $L \geq 0$ and $Le^T \leq 1$
- An upper bound on the transition probabilities $U \in \mathbb{R}^{n \times n}$ with $U \geq L$ and $Ue^T \geq 1$

- Goal: Computation of bounds for $pr$
  where $pP = p$, $pe^T = 1$ and $P$ is an irreducible stochastic matrix with $L \leq P \leq U$
Basic Definitions

Reasons for knowing only bounds on transition probabilities:

- Limited knowledge of the application and its environment
- Limited accuracy of measurements
- Abstraction to achieve Markovian behavior
- Markov chains resulting from decomposition and aggregation

Approach can be extended to continuous time Markov chains with

- Bounds on the transition rate matrix $V \leq W$ such that
  - $V(i,j), W(i,j) \geq 0$ for $i \neq j$ and $V(i,i) \leq -\sum_{j \neq i} V(i,j)$, $W(i,i) \leq -\sum_{j \neq i} W(i,j)$

by using randomization

(i.e., $L = V/\alpha + I$ and $U = W/\alpha + I$, $\alpha \geq \max(|V(i,i)|)$)

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Basic Definitions

Definition of sets of matrices

- $\mathcal{P}_L = \{ P \mid P \geq L, P e^T = e^T \text{ and } P \text{ irreducible} \}$
  - $\mathcal{V}_L = \{ v \mid v \geq 0, \exists P \in \mathcal{P}_L: vP = v \text{ and } ve^T = 1 \}$

- $\mathcal{P}_U = \{ P \mid P \leq U, P e^T = e^T \text{ and } P \text{ irreducible} \}$
  - $\mathcal{V}_U = \{ v \mid v \geq 0, \exists P \in \mathcal{P}_U: vP = v \text{ and } ve^T = 1 \}$

- $\mathcal{P}_{L,U} = \mathcal{P}_L \cap \mathcal{P}_U$
  - $\mathcal{V}_{L,U} = \{ v \mid v \geq 0, \exists P \in \mathcal{P}_{L,U}: vP = v \text{ and } ve^T = 1 \}$

Find bounds $\min_{v \in \mathcal{V}_{L,U}} (vr)$ and $\max_{v \in \mathcal{V}_{L,U}} (vr)$

- Bounds on the expected reward knowing bounds on transition probabilities/rates

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Courtois/Semal Bounds

Bounds based on $P_L$ (Courtois/Semal JACM 1984)

- Let $N = (I - L)^{-1}$ and $Z = (\text{diag}(Ne^T))^{-1}N$
  - $N \geq 0$ exists and $Z$ is a stochastic matrix
  - $\min_{v \in V_L} (vr) = \min_{i \in \{1, \ldots, n\}} (e_iZr)$ and
    $\max_{v \in V_L} (vr) = \max_{i \in \{1, \ldots, n\}} (e_iZr)$

Bounds based on $P_U$ (Courtois/Semal JACM 1984)

- Let $M = (I - U)^{-1}$ and $Y = (\text{diag}(Me^T))^{-1}M$
  - If $M$ exists and $Me^T < 0$, $eM < 0$, then $Y$ is a stochastic matrix and
    - $\min_{v \in V_U} (vr) = \min_{i \in \{1, \ldots, n\}} (e_iYr)$ and
      $\max_{v \in V_U} (vr) = \max_{i \in \{1, \ldots, n\}} (e_iYr)$
Courtois/Semal Bounds

- Best possible bounds if only $P_L$ or $P_U$ are known (but not the best bounds if $P_{L,U}$ is known!)
- Extensions and improvements of the approach until recently
  - Better bounds and alternative bound computation if the method is used in combination with aggregation e.g. Franceschinis/Muntz 1984, Semal 1995
  - Improved bounds, partially based on $P_{L,U}$ for upper Hessenberg matrices e.g. Muntz/Lui 1994, Mahevas/Rubino 2001
- A new approach for bound computation based on $P_{L,U}$ that is applicable for arbitrary matrices
Courtois/Semal Bounds

An example to clarify the problem:
Mean population/blocking prob. in a M/M/1/K system with \( \lambda \in [\lambda^-, \lambda^- + \varepsilon_1] \) and \( \mu \in [\mu^-, \mu^- + \varepsilon_2] \)
Results can be computed only from \( L \) not from \( U \) for our parameters!

Lower bounds of transition rates
(diagonal elements are not printed)

\[
\begin{pmatrix}
-\Sigma & \lambda^- \\
\mu^- & -\Sigma & \lambda^- \\
& \ddots & \ddots & \ddots \\
& & \mu^- & -\Sigma & \lambda^- \\
& & & \mu^- & -\Sigma
\end{pmatrix}
\]
Courtois/Semal Bounds

An example to clarify the problem:
Mean population/blocking prob. in a M/M/1/K system with $\lambda \in [\lambda^-, \lambda^-+\varepsilon_1]$ and $\mu \in [\mu^-, \mu^-+\varepsilon_2]$.
Results can be computed only from $L$ not from $U$ for our parameters!

Matrix yielding the lower bound for the rewards
(diagonal elements are not printed)

\[
\begin{pmatrix}
-\Sigma & \lambda^- \\
\mu^- + \varepsilon_1 + \varepsilon_2 & -\Sigma + \lambda^- \\
\vdots & \vdots & \vdots & \vdots \\
\varepsilon_1 + \varepsilon_2 & \mu^- & -\Sigma & \lambda^- \\
\varepsilon_1 + \varepsilon_2 & \mu^- & -\Sigma \\
\end{pmatrix}
\]
Courtois/Semal Bounds

An example to clarify the problem:
Mean population/blocking prob. in a M/M/1/K system with $\lambda \in [\lambda^-, \lambda^-+\varepsilon_1]$ and $\mu \in [\mu^-, \mu^-+\varepsilon_2]$ Results can be computed only from $L$ not from $U$ for our parameters!

Matrix yielding the upper bound for the rewards (diagonal elements are not printed)

$$
\begin{pmatrix}
-\Sigma & \lambda^- & \varepsilon_1 + \varepsilon_2 \\
\mu^- & -\Sigma & \lambda^- & \varepsilon_1 + \varepsilon_2 \\
\cdots & \cdots & \cdots & \cdots \\
\mu^- & -\Sigma & \lambda^- + \varepsilon_1 + \varepsilon_2 \\
\mu^- & -\Sigma
\end{pmatrix}
$$

Matrices for the bounds include transitions changing the population in the system by adding/removing K customers in one step $\Rightarrow$ impossible from the system specification!!
Improved Bounds

Matrices yielding better bounds (in this case easy to prove)

Matrix yielding the lower bound for the rewards
(diagonal elements are not printed)

\[
\begin{pmatrix}
-\Sigma & \lambda^- \\
\mu^- + \varepsilon_2 & -\Sigma & \lambda^-
\end{pmatrix}
\]

Matrix yielding the upper bound for the rewards
(diagonal elements are not printed)

\[
\begin{pmatrix}
-\Sigma & \lambda^- + \varepsilon_1 \\
\mu^- & -\Sigma & \lambda^- + \varepsilon_1
\end{pmatrix}
\]
Improved Bounds

Mean population for varying \( K \)
\( (\lambda^- = 0.475, \; \varepsilon_1 = 0.05, \; \mu^- = 0.95, \; \varepsilon_2 = 0.1) \)
Improved Bounds

Blocking probability for varying $\lambda^-$
($K = 20$, $\varepsilon_1 = 0.1 \cdot \lambda^-$, $\mu^- = 0.95$, $\varepsilon_2 = 0.1$)
Improved Bounds

If $L \neq U$, then $P_{L,U}$ is an infinite set!

For the computation of bounds we need to generate two matrices

- $P^- \in P_{L,U}$ with steady state vector $v^-$ such that $v^- r = \min_{v \in V_{L,U}} (vr)$

- $P^+ \in P_{L,U}$ with steady state vector $v^+$ such that $v^+ r = \max_{v \in V_{L,U}} (vr)$

How to compute these matrices from an infinite set?
Improved Bounds

A matrix $P \in P_{L,U}$ is an extremal point of $P_{L,U}$ iff no two matrices $P_1, P_2 \in P_{L,U} (P_1 \neq P_2)$ exist such that $P = \beta P_1 + (1-\beta)P_2$ for $0 < \beta < 1$

$E_{L,U} = \{P | P \in P_{L,U} \text{ and } P \text{ is extremal point of } P_{L,U}\}$

**Theorem 1:**
1. $P \in E_{L,U} \iff \forall r \in \{1, \ldots, n\} \exists c_r \in \{1, \ldots, n\}$ such that $P(r,i) = L(r,i)$ or $P(r,i) = U(r,i)$ for all $i \neq c_r$
2. The set $E_{L,U}$ is finite and contains at most $(n!)^n$ matrices
Improved Bounds

**Theorem 2:**

1. \( \exists P^- \in E_{L,U} \) with steady state vector \( \mathbf{v}^- \) such that
   \[ \mathbf{v}^- \mathbf{r} = \min_{\mathbf{v} \in V_{L,U}} (\mathbf{v} \mathbf{r}) \]

2. \( \exists P^+ \in E_{L,U} \) with steady state vector \( \mathbf{v}^+ \) such that
   \[ \mathbf{v}^+ \mathbf{r} = \max_{\mathbf{v} \in V_{L,U}} (\mathbf{v} \mathbf{r}) \]

Where are we?

- The set can be enumerated and the maximum/minimum can be computed
- Unfortunately, the effort is too high even for small matrices
A new Bounding Algorithm

- Let $\Pi$ be the set of permutations on $\{1,\ldots,n\}$
- For $\pi \in \Pi$ generate a matrix $P$ as follows in at most $n$ steps:
  - Start with $P^{(0)} = L$
  - Let for $m=1,\ldots,n$, $i=1,\ldots,n$ and $j=\pi(m)$:
    \[
    P^{(m)}(i,j) = \min(U(i,j), 1 - \sum_{k \neq i} P^{(m-1)}(i,k))
    \]
- Let $F_{L,U}$ be the set of all matrices $P$ which are generated according to some $\pi \in \Pi$

**Theorem 3:**

1. $F_{L,U} \subseteq E_{L,U}$
2. $F_{L,U}$ contains at most $n!$ matrices
3. $P, P^+ \in F_{L,U}$

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A new Bounding Algorithm

For $\pi \in \Pi$ define

$$\Pi^{(1)}(\pi) = \{ \theta \mid \exists i,j: \pi(i)=\theta(j), \pi(j)=\theta(i) \text{ and } \forall k \neq i,j: \pi(k)=\theta(k) \}$$

- $\Pi^{(1)}(\pi)$ contains at most $n(n-1)/2$ permutations

**Theorem 4:**
Let $P_{\pi} \in F_{L,U}$ be generated using permutation $\pi$ and let $v_{\pi}$ the corresponding stationary vector, then

1. if $v_{\pi}r \leq v_{\theta}r$ for all $\theta \in \Pi^{(1)}(\pi) \Rightarrow v_{\pi}r = \min_{v \in V_{L,U}} (vr)$

2. if $v_{\pi}r \geq v_{\theta}r$ for all $\theta \in \Pi^{(1)}(\pi) \Rightarrow v_{\pi}r = \max_{v \in V_{L,U}} (vr)$
A new Bounding Algorithm

Outline of an algorithm (here computation of the upper bound):

1. Choose one $\pi \in \Pi$ and compute $P_{\pi}$
2. While $\theta \in \Pi^{(1)}(\pi)$ exist such that $v_{\pi}r \leq v_{\theta}r$ do
   i. $\pi = \theta$
3. $v_{\pi}r$ is the upper bound for the reward

Effort of the approach

- Analysis of a new permutation in step 2 requires rank one update of an inverse matrix for all columns (worst case effort $O(n^3)$)
- Number of permutations $\pi$ that is considered can be in $O(n!)$

Worst case complexity is horrible, but average case seems to be much better!
A new Bounding Algorithm

Capacity 20

\[ \lambda \in [0.792, 0.808] \]
\[ \mu \in [0.990, 1.010] \]
\[ \xi \in [0.99 \cdot 10^{-x}, 1.01 \cdot 10^{-x}] \]
\[ \nu \in [0.99 \cdot 10^{-x-1}, 1.01 \cdot 10^{-x-1}] \]

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A new Bounding Algorithm

- Erlang 2 distribution at the central station
- $K$ nearly lumpable peripheral stations
- $N$ customers

State space detailed/aggregated ($K,N$)

Spread of the bounds for the throughput

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Conclusions and Outlook

✓ Bounding matrices in a polyhedron of matrices
✓ Algorithm to compute bounding matrices
✓ Bounds are sharp (and improve known bounds significantly)
✓ Effort for bound computation is high

➢ Further Investigation of the algorithm
➢ Application of the approach for decomposition and aggregation
➢ More efficient methods for specific classes of matrices