Numerical Analysis of Generalized Semi-Markov Processes

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Abstract

This paper present methodological results that allow the cost-effective numerical analysis of finite-state generalized semi-Markov processes (GSMPs) with exponential and deterministic events by an embedded general state space Markov chain (GSSMC). Key contributions constitute the formal proof that elements of the transition kernel of the GSSMC can always be computed by appropriate summation of transient state probabilities of continuous-time Markov chains and the derivation of conditions under which kernel elements are constant. Furthermore, we derive conditions on the building blocks of the GSMP for which state probabilities $\pi_i(a_1, a_2)$ are symmetric in respect to clock readings of deterministic events concurrently active. The exploitation of these properties is the key driver to the cost-effective time-dependent and stationary analysis of the considered class of GSMPs. The techniques of this paper are applicable to networks of queues, stochastic Petri nets, time-enhanced state charts and UML specifications, and other discrete-event stochastic systems with an underlying stochastic process that can be represented as a GSMP with exponential and deterministic events.

Keywords: Techniques and algorithms for stochastic modeling,
discrete-event stochastic systems with deterministic delays,
general state space Markov chains,
numerical transient analysis of continuous-time Markov chains,
Fredholm integral equations with separable kernel.

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1 Introduction

Since many activities associated with computer and communication systems have a constant duration, performance and dependability models of such systems should allow representation of both stochastic and deterministic timing. Activities of computer systems which have a constant duration include memory access times, transfer times for data packets of fixed size, time-outs, and repair times of components. This paper deals with numerical methods for analysis of discrete-event systems with stochastic and deterministic timing. A discrete-event stochastic system makes state transitions when events associated with the occupied state occur; events occur only at an increasing sequence of random times. The underlying stochastic process of a discrete-event stochastic system records the state of the system as it evolves over continuous time. The usual model for this process is a generalized semi-Markov process (GSMP); see e.g., Glasserman and Yao [5], Glynn [6], Shedler [11], and Whitt [12],

In this paper, we present methodological results that allow the cost-effective numerical analysis of finite-state generalized semi-Markov processes (GSMPs) with exponential and deterministic events by an embedded general state space Markov chain (GSSMC). Lindemann and Shedler introduced a GSSMC embedded at equidistant time points nD (n=1,2,..) of the continuous-time GSMP and showed that both the GSMP and the GSSMC have the same stationary or time-averaged distributions [9]. Numerical solvers for the system of multidimensional Fredholm integral equations that constitute the time-dependent and stationary equations of the GSSMC have recently been presented [10].

To make this GSSMC approach effectively applicable in performance and dependability modeling projects at large, the remaining open problem constitutes the algorithmic generation of the simplest form of the transition kernel of this GSSMC given the building blocks of the GSMP. The transition kernel of the GSSMC specifies one-step jump probabilities from a given state at instant of time nD to all reachable new states at instant of time (n+1)D. In general, elements of the transition kernel of a GSSMC are functions of clock readings associated with the current state and intervals for clock readings associated with the new state.

This paper presents three theorems that provide the foundation for such an algorithmic generation of the transition kernel. Key contributions constitute the formal proof that kernel elements can always be computed by summation of transient state probabilities of continuous-time Markov chains (Theorem 1) and the derivation of conditions on the building blocks of the GSMP under which kernel elements are constant; i.e., are not functions of clock readings (Theorem 2). Furthermore, we derive conditions on the building blocks of the GSMP for which state probabilities $\pi_i(a_1, a_2)$ are symmetric in respect to clock readings of deterministic events concurrently active (Theorem 3). That is $\pi_i(a_1, a_2) = \pi_i(a_2, a_1)$. The exploitation of these properties of the GSSMC considerably reduces the computing time and memory requirements for the numerical solution of the system of Fredholm integral equations. Thus, the presented methodology constitutes the key driver for the cost-effective numerical analysis of GSMPs with large state space (e.g., 100,000 states) and several deterministic events concurrently active. The techniques of this paper are applicable to networks of queues, deterministic and stochastic Petri nets [1], stochastic process algebras [2], stochastic automata networks [6], time-enhanced state charts and UML specifications, and other discrete-event stochastic systems with an underlying stochastic process that can be represented as a GSMP with exponential and deterministic events.

The remainder of this paper is organized as follows. In Section 2 we show how to define the GSSMC underlying a GSMP with exponential and deterministic events and introduce the notation. Section 3 first recalls the form of the transition kernel. Then, we prove three theorems on properties of the GSSMC. The exploitation of these properties is key to the cost-effective numerical solution of the systems of Fredholm integral equations representing the time-dependent and stationary equations of the considered class of GSMPs. In Section 4, we illustrate the impact of these theorems. Finally, concluding remarks are given.

2 Derivation of the Embedded General State Space Markov Chain

A generalized semi-Markov process (GSMP) is a continuous-time stochastic process $\{S(t): t \geq 0\}$ that makes a state transition when one or more "*events*" associated with the occupied state occur. Events associated with a state compete to trigger the next state transition, and each set of trigger events has its own distribution for determining the next state. At each state transition of the GSMP, *new* events may be scheduled. For each of these new events, a clock indicating the time until the event is scheduled to occur is set according to an independent (stochastic) mechanism. I.e., for each new event a clock reading is generated according to its *clock setting distribution*. For each scheduled event which does not trigger a state transition but is still scheduled in the next state, its clock *continues* to run. If an event is no longer scheduled in the next state, it is *canceled*, and the corresponding clock reading is discarded. In general, in a GSMP events may occur simultaneously resulting in a set of trigger events E* rather than in a unique trigger event e* [11].

Let $E = \{e_1, e_2, ..., e_K\}$ be a finite set of events and S be a finite set of states. For a state $s \in S$, let $s \mapsto E(s)$ be a mapping from the set S to the nonempty subsets of E; E(s) denotes the set of all events that are scheduled to occur when the process is in state s. When the process is in state s, the (simultaneous) occurrence of one or more events of E(s) triggers a state transition to a state s'. Denote the probability that the new state is s' given that the event e* and the set of Events E* occur in state s by $p(s', s, e^*)$ and $p(s', s, E^*)$, respectively. For each s $\in$ S and e* $\in$ E(s) or E* $\subseteq$ E(s), we assume that $p(\cdot; s, e^*)$ or $p(\cdot; s, E^*)$ is a probability mass function. Associated with each event is a clock with a reading that records the remaining time until the event is scheduled to trigger a state transition. For $s_i \in S$, define the set C(i) of possible clock-reading vectors in state s_i as:

$$C(i) = \{(c_1, \ldots, c_K): c_k \geq 0 \text{ and } c_k > 0 \text{ if and only if } e_k \in E(i)\} \tag{1}$$

The k-th component of a clock-reading vector $\mathbf{c} = (c_1, c_2, ..., c_K)$ is the clock reading associated with event e_k.

In this paper, we consider finite-state, time-homogeneous GSMPs with exponential and deterministic clock setting distributions. We divide the set of events $E = E_{exp} \cup E_{det}$ and enumerate the deterministic events by $e_1, e_2, \ldots, e_M$. Subsequently, we define D_m to be the firing delay of event e_m $(1 \leq m \leq M)$. For the analysis of this class of GSMPs, in [9] a discrete-time general state space Markov chain (GSSMC) has been introduced. According to [9], we define $D = \min\{D_1, D_2, \ldots, D_M\}$. To derive this GSSMC, we define a discrete-time process $X_n = \{X(nD): n \geq 0\}$ by observing the GSMP $\{S(t): t \geq 0\}$ at a sequence $\{nD: n \geq 0\}$ of fixed times

$$X_n = (S_n, C_n(1), C_n(2), \ldots, C_n(M)) \tag{2}$$

Here, S_n represents the state of the GSMP and $C_n(m)$ represents the clock reading of deterministic event e_m ($1 \le m \le M$) at instant of time nD. When deterministic event e_m is not active at time nD, we set $C_n(m) = 0$. The memoryless property of the exponential distribution implies that $\{X(nD): n \ge 0\}$ is a GSSMC, i.e., it satisfies the Markov property. That is:

$$P\left[X_{n+1} \in \mathcal{A} \middle| X_n, X_{n-1}, \ldots, X_0\right] = P\left[X_{n+1} \in \mathcal{A} \middle| X_n\right]$$

for any appropriately defined set $\mathcal{A}$. For ease of exposition, we restrict the discussion to GSMPs in which at most two deterministic events may be concurrently enabled. However, we would like to point out that the theorems presented in Section 3 can be extended in a straightforward way for GSMPs with more than two deterministic events concurrently active.

The subset of states of the GSSMC in which only exponential events are enabled is denoted by S_{exp}. Similarly, the subsets of states in which one deterministic event and two deterministic events are (concurrently) enabled are denoted by S_{det1} and S_{det2}, respectively. Subsequently, without loss of generality, we enumerate the states of the GSMP as follows:

$$\begin{aligned} S_{exp} &= \{s_1, s_2, \ldots, s_{N_1}\} \\ S_{det1} &= \{s_{N_1+1}, s_{N_1+2}, \ldots, s_{N_1+N_2}\} \\ S_{det2} &= \{s_{N_1+N_2+1}, s_{N_1+N_2+2}, \ldots, s_N\} \end{aligned} \tag{3}$$

We denote the index of the deterministic event(s) enabled in a state s_i by l(i) and m(i), respectively, and neglect other zero-valued clock readings in C(i). Given the initial distribution of the GSMP, denoted by X_0 and using (2), we define for the GSSMC X_n with two deterministic events concurrently enabled three kinds of time-dependent state probabilities:

$$\begin{aligned} &\pi_i^{(n)} = P\{S_n = s_i \mid X_0\} && \text{for } s_i \in S_{exp} \\ &\pi_i^{(n)}(a_1) = P\{S_n = s_i, C_n(l(i)) \le a_1 \mid X_0\} && \text{for } s_i \in S_{det1} \\ &\pi_i^{(n)}(a_1, a_2) = P\{S_n = s_i, C_n(l(i)) \le a_1, C_n(m(i)) \le a_2 \mid X_0\} && \text{for } s_i \in S_{det2} \end{aligned} \tag{4}$$

for n = 1,2,... and $0 < a_1, a_2 \le D$.

Subsequently, the transient state probabilities of the GSMP at instants of time t = nD are given by $\pi_i^{(n)}$ for $s_i \in S_{exp}$, $\pi_i^{(n)}(D_{l(i)})$ for $s_i \in S_{det1}$ and, $\pi_i^{(n)}(D_{l(i)}, D_{m(i)})$ for $s_i \in S_{det2}$, respectively. Corresponding stationary or time-averaged state probabilities of the GSMP are denoted as π_i, $\pi_i(D_{l(i)})$, and $\pi_i(D_{l(i)}, D_{m(i)})$.

3 Theorems on Properties of the General State Space Markov Chain

3.1 General Form of the Transition Kernel

A GSSMC is completely specified by a transition kernel (heuristically, this is a family of probability matrices) and an initial distribution at time t = 0. The transition kernel of the

GSSMC specifies one-step jump probabilities from a given state at instant of time nD to all reachable new states at instant of time (n+1)D. As for an ordinary discrete-time Markov chain, for all states s_j not reachable from s_i corresponding jump probabilities $p_{ij}(.)$ are zero. In general, elements of the transition kernel of a GSSMC are functions of clock readings associated with the current state s_i and the new state s_j. In this section, we present three theorems on properties of the transition kernel of the GSSMC.

The transition kernel of the GSSMC $X_n = \{(S_n, \mathbf{C}_n) \colon n \geq 0\}$ constitutes a functional matrix of the form $\mathbf{P}(\mathbf{c}, \mathbf{A})$. In general, the elements of the transition kernel $\mathbf{P}(\mathbf{c}, \mathbf{A})$ of the GSSMC have the form:

$$p_{ij}(\mathbf{c}, \mathbf{A}) = P\left\{X_{n+1} \in \{s_j\} \times \mathbf{A} \,\middle|\, X_n = (s_i, \mathbf{c})\right\} \tag{5}$$

Restricting the discussion to GSMPs with at most two deterministic events concurrently active, the vector of old clock readings **c** and the set **A** for intervals of new clock readings are given by:

$$\mathbf{c} = \mathbf{c}(s_i) = \begin{cases} \varnothing & , s_i \in S_{exp} \\ c_1 & , s_i \in S_{det1} \\ (c_1, c_2) & , s_i \in S_{det2} \end{cases} \quad \text{and} \quad \mathbf{A} = \mathbf{A}(s_j) = \begin{cases} \varnothing & , s_j \in S_{exp} \\ (0, a_1] & , s_j \in S_{det1} \\ (0, a_1] \times (0, a_2] & , s_j \in S_{det2} \end{cases} \tag{6}$$

Thus, for GSMPs with at most two deterministic events concurrently active, the transition kernel of the GSSMC can be expressed by a functional matrix $\mathbf{P}(c_1,c_2,a_1,a_2)$. Subsequently, an element of this kernel $p_{ij}(.)$ is in general a function in four variables c_1, c_2, a_1, and a_2. However, we will observe that a large number of kernel elements are constant; i.e., $p_{ij}(c_1,c_2,a_1,a_2) = p_{ij}$. Furthermore, for most functional kernel elements new clock readings need not be considered; i.e., $p_{ij}(c_1,c_2,a_1,a_2) = p_{ij}(c_1,c_2)$.

$$\mathbf{P}(c_1,c_2,a_1,a_2) = \begin{array}{cc}
\left(\begin{array}{c|c|c}
\mathbf{P}_{11} & \mathbf{P}_{12}(a_1) & \mathbf{P}_{13}(a_1,a_2) \\ \hline
\mathbf{P}_{21}(c_1) & \mathbf{P}_{22}(c_1,a_1) & \mathbf{P}_{23}(c_1,a_1,a_2) \\ \hline
\mathbf{P}_{31}(c_1,c_2) & \mathbf{P}_{32}(c_1,c_2,a_1) & \mathbf{P}_{33}(c_1,c_2,a_1,a_2)
\end{array}\right) &
\begin{array}{c} 1 \\ \vdots \\ N_1 \\ \hdashline N_1+1 \\ \vdots \\ N_1+N_2 \\ \hdashline N_1+N_2+1 \\ \vdots \\ N \end{array} \\
\begin{array}{c:c:c} 1 \qquad\qquad N_1 & N_1+1 \quad N_1+N_2 & N_1+N_2+1 \quad N \end{array} &
\end{array} \tag{7}$$

Equation (7) shows the general form of the kernel $\mathbf{P}(c_1,c_2,a_1,a_2)$ as a composition of nine submatrices $\mathbf{P}_{ij}(.)$ of appropriate dimension using (5) and (6). In (7), the submatrix $\mathbf{P}_{11}$ represents state transition among states of S_{exp}, $\mathbf{P}_{12}(a_1)$ represents state transitions from states of S_{exp} to states of S_{det1}, and $\mathbf{P}_{13}(a_1,a_2)$ represents state transitions from states of S_{exp} to states of S_{det2}. Furthermore, submatrix $\mathbf{P}_{22}(c_1,a_1)$ represents state transitions among states of S_{det1} and $\mathbf{P}_{21}(c_1)$ represents state transitions from states of S_{det1} to states of S_{exp}. The submatrices $\mathbf{P}_{23}(c_1,a_1,a_2)$ represents state transitions from states of S_{det1} to states of S_{det2}, respectively. State transitions from states of S_{det2} to states of S_{det1} and S_{exp} are represented by the submatrices $\mathbf{P}_{32}(c_1,c_2,a_1)$ and $\mathbf{P}_{31}(c_1,c_2)$. The submatrix $\mathbf{P}_{33}(c_1,c_2,a_1,a_2)$ represents state transitions among states of S_{det2}.

For kernel elements of $\mathbf{P}_{33}(c_1,c_2,a_1,a_2)$, in general, there may exist 24 possible orderings for clock readings c_1, c_2, a_1, and a_2. These orderings immediately lead to the regions of integration in the system of integral equations presented in Section 4.1. Figure 1 shows the two possible orderings for kernel elements in $\mathbf{P}_{31}(c_1,c_2)$ and the six possible subregions for elements in $\mathbf{P}_{32}(c_1,c_2,a_1)$. Finally, Figure 2 shows the 24 subregions for kernel elements in $\mathbf{P}_{33}(c_1,c_2,a_1,a_2)$. However, when deterministic events $e_{l(i)}$ and $e_{m(i)}$ associated with state s_i cannot be canceled, only the following eight orderings of clock readings may occur:

(1) $0 < c_1 < c_2 \le a_1 \le a_2$　　(2) $0 < c_2 < c_1 \le a_1 \le a_2$

(3) $0 < c_1 \le a_1 < c_2 \le a_2$　　(4) $0 < c_2 \le a_1 < c_1 \le a_2$

(5) $0 < c_1 < c_2 \le a_2 \le a_1$　　(6) $0 < c_2 < c_1 \le a_2 \le a_1$

(7) $0 < c_1 \le a_2 < c_2 \le a_1$　　(8) $0 < c_2 \le a_2 < c_1 \le a_1$

Furthermore, if state probability $\pi_i(a_1,a_2)$ is symmetric with respect to a_1 and a_2 and the deterministic events $e_{l(i)}$ and $e_{m(i)}$ cannot be canceled, only the orderings (1) to (4) have to be considered. Note that within each region shown in Figure 1 and 2, functional kernel elements are continuous and differentiable. However, in general, functional kernel elements $p_{ij}(.)$ are

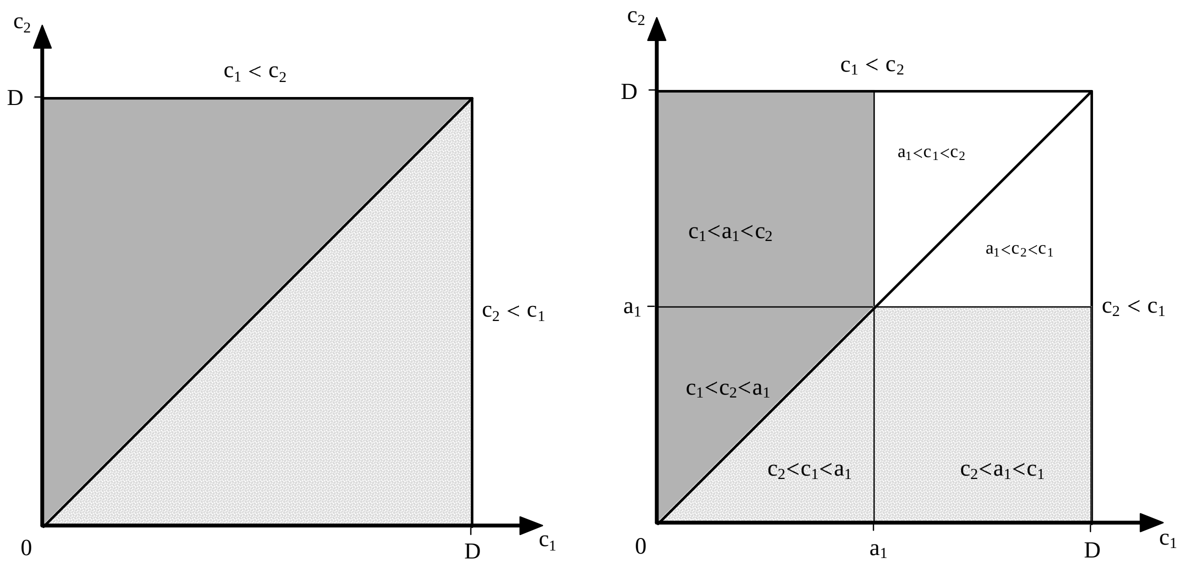

Figure 1. Regions of integration in integral equations for π_{exp} and $\pi_{det1}(a_1)$

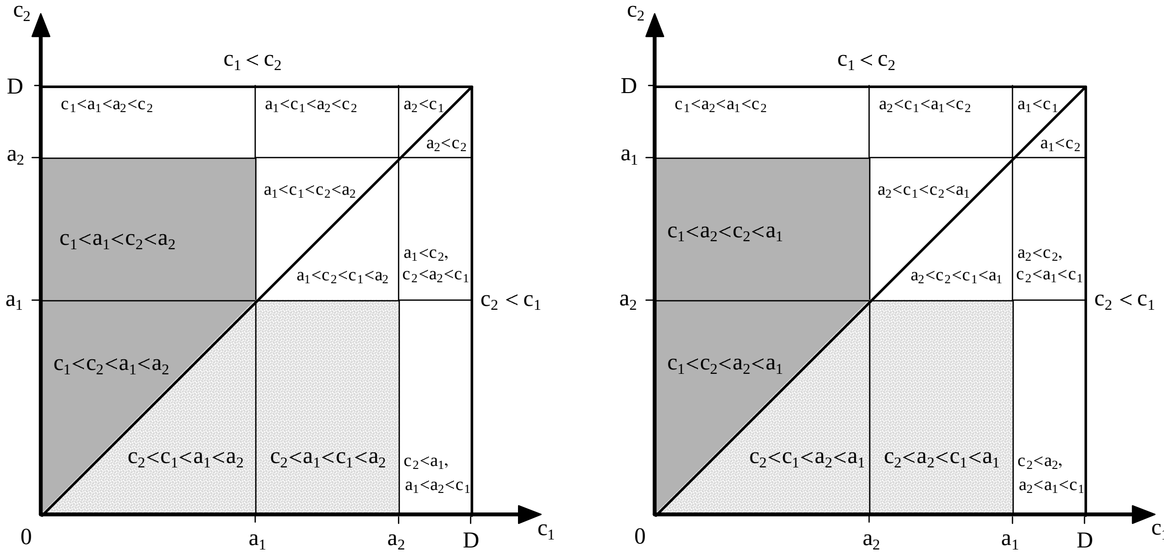

Figure 2. Regions of integration in equations for $\pi_{det2}(a_1,a_2)$ with $a_1 \le a_2$ and with $a_2 \le a_1$

not differentiable at the boundary. This is because different orderings of clock readings may lead to different functional kernel elements $p_{ij}(.)$.

3.2 Numerical Computation of the Transition Kernel

In [8], the concept of subordinated Markov chains (SMCs) has been introduced for the efficient algorithmic computation of the probability matrix P of the discrete-time Markov chain embedded in the Markov regenerative process underlying a discrete-event stochastic system without concurrent deterministic events. The SMC of state si is a continuous-time Markov chain (CTMC) whose state space is given by the transitive closure of all states reachable from si via a (possible empty) sequence of exponential events and corresponding next state probabilities $p(s', s, e^*)$ of the GSMP. For such a sequence of exponential events from s_i to s_j, we write $s_i \xrightarrow{\exp^*} s_j$.

We can define a SMC for each state of the GSMP, i.e., also for states in which only exponential events are active.

Definition (Subordinated Markov chain and subordinated reachability set):

The continuous-time Markov chain $\{X_i(t)\colon t \ge 0\}$ with state spaces $\mathrm{SMC}(s_i) = \{s \in S \mid s_i \xrightarrow{\exp^*} s\}$ and state transitions corresponding to the occurrence of exponential events is called the *subordinated Markov chain (SMC)* of state s_i. The generator matrix of $\mathrm{SMC}(s_i)$ is denoted by $\mathbf{Q}_i$. The initial distribution is $P\{X_i(0) = s_i\} = 1$.

Furthermore, we define the set of states from which a state s_j is only reachable via the occurrence of exponential events as the *subordinated reachability set (SRS)* of state s_j. Formally, that is $\mathrm{SRS}(s_j) = \{s \in S \mid s \xrightarrow{\exp^*} s_j\}$

The probability for a state transition from state s_i to state s_j in time t via the occurrence of only exponential events is given by:

$$\tilde{p}_{s_i,s_j}(t) \overset{\text{def}}{=} P\left\{s_i \xrightarrow[t]{\exp^*} s_j\right\} = P\left\{X_i(t) = s_j \;\middle|\; X_i(0) = s_i\right\} = \mathbf{1}_i^T \cdot e^{\mathbf{Q}_i t} \cdot \mathbf{1}_j \qquad (11)$$

where $\mathbf{1}_i^T$ and $\mathbf{1}_j$ denote the i-th and j-th row and column unity-vectors, respectively, of appropriate dimension. Using the randomization technique [7], transient state probabilities of SMCs of (11) can computed with asymptotical effort $O(\eta_i q_i D)$ for all time points t = Δ, 2Δ,,..., MΔ = D. Here, η_i denotes the number of nonzero entries in the generator $\mathbf{Q}_i$ and q_i the absolute value of its maximum diagonal entry.

The following provides an intuitive explanation why elements of the transition kernel of a GSSMC can always be determined by appropriate sums of transient state probabilities of continuous-time Markov chains. Assuming the GSMP is at time nD in state s_i with two deterministic events $e_{l(i)}$ and $e_{m(i)}$ concurrrently active. Thus, the GSSMC resides in a state, say (s_i,c_1,c_2) with $c_1 \le c_2$, where c_1 and c_2 are clock readings associated with deterministic events $e_{l(i)}$ and $e_{m(i)}$, respectively. Noting that the state of the GSMP at time (n+1)D given the state at time nD is determined by (possibly empty) sequences of exponential events in the subintervals $((nD, nD+c_1]$, $(nD+c_1, nD+c_2]$ and $(nD+c_2, (n+1)D]$ and the occurrence of the deterministic events e_k and e_m. Thus, using the property that the GSMP is time-homogeneous and by decomposing the time interval (0,D] into three subintervals $(0,c_1]$, $(c_1,c_2]$, and $(c_2,D]$, we can show that the GSMP behaves in each subinterval as a CTMC. Each of these three CTMCs is given by an SMC as defined above. Subsequently, the kernel elements of the embedded GSSMC can be computed as summations of transient state probabilities of SMCs. It is important to note that this holds irrespective of the number of deterministic events active in states s_i and s_j.

The following theorem states how the transition kernel of the GSSMC can be effectively computed and constitutes the main result of the paper.

Theorem 1 (Numerical computation of the transition kernel)

Let $\{X(t): t \ge 0\}$ be a finite state space GSMP with exponential and deterministic events. Then all elements $p_{ij}(.)$ of the transition kernel $\mathbf{P}(c_1,c_2,a_1,a_2)$ of the embedded GSSMC $\{X_n: n \ge 0\}$ can be computed simply by the summation of transient state probabilities of continuous time Markov chains.

Proof: We prove this result by construction. A complete proof of Theorem 1 requires the consideration of nine different forms of kernel elements introduced in Eq. (7) and 24 orderings of clock readings for $\mathbf{P}_{33}(c_1,c_2,a_1,a_2)$ as shown in Figure 2. Due to space limitations, we spell out the derivation only for five selected forms, namely for the submatrices $\mathbf{P}_{11}$, $\mathbf{P}_{12}(a_1)$, $\mathbf{P}_{13}(a_1,a_2)$, $\mathbf{P}_{22}(c_1,a_1)$, and one ordering for $\mathbf{P}_{33}(c_1,c_2,a_1,a_2)$. Kernel elements of other forms can be derived in a similar way . Recall that $X_n = \{(S_n, \mathbf{C}_n): n \geq 0\}$ is the GSSMC that is embedded in a GSMP at equidistant time points nD.

We write $z \xrightarrow{e_m} z'$ for the state transition from state z to z' due to the occurrence of deterministic event e_m. In (12) to (18), we write (z, z', e_m) as shorthand index of a summation over all feasible paths of the form $z \xrightarrow{e_m} z'$. For ease of exposition, we assume $p(z', z, e_m) = 1$ in the proof of Theorem 1. The extension of Eqs. (15) to (18) to an arbitrary pmf of next state probabilities $p(z', z, E^*)$ is straight-forward; i.e., requires one additional summation. In the following, we denote by $S(e_i) \subseteq S_{det1} \cup S_{det2}$ all states in which deterministic event e_i is active. By $S(e_i, e_j) \subseteq S_{det2}$, we denote all states in which both deterministic events e_i and e_j are active.

Let us first derive how to compute kernel elements of the submatrix $\mathbf{P}_{11}$. Since no deterministic event is active in state s_i, we need not decompose the time interval (0,D] into subintervals. Deterministic events cannot have triggered a state transition in (0,D], though, some deterministic events may have become active and get canceled in (0,D]. Thus, considering the SMC of state s_i and using (11), we derive the kernel element as:

$$p_{ij} = P\left\{s_i \xrightarrow[D]{\exp^*} s_j\right\} = \tilde{p}_{s_i, s_j}(D) \tag{12}$$

That is the probability for a state transition from s_i to s_j. Eq. (12) implies that kernel elements $p_{ij} = 0$ for states $s_j \notin SMC(s_i)$.

Now consider the derivation of kernel elements of the submatrix $\mathbf{P}_{12}(a_1)$. Corresponding kernel elements are of the form $p_{ij}(a_1)$ where a_1 denotes the boundary of the clock reading interval of the deterministic event newly active; see Eq. (6). Subsequently, we decompose the time interval (0, D] into subintervals (0, a_1] and (a_1, D]. Using (11), we have:

$$\begin{aligned} p_{ij}(a_1) &= P\left\{S_{n+1} = s_j, C_{n+1}(l(j)) \leq a_1 \;\middle|\; S_n = s_i\right\} \\ &= \sum_{z \in Z} P\left\{s_i \xrightarrow[a_1]{\exp^*} z\right\} \cdot P\left\{z \xrightarrow[D-a_1]{\exp^*} s_j\right\} = \sum_{z \in Z} \tilde{p}_{s_i, z}(a_1) \cdot \tilde{p}_{z, s_j}(D - a_1) \end{aligned} \tag{13}$$

with the set of intermediate states $Z = S_{det1} \cap SMC(s_i) \cap SRS(s_j)$.

Subsequently, we consider the derivation of kernel elements of the submatrix $\mathbf{P}_{13}(a_1,a_2)$. Recall that a_1 and a_2 denote upper bounds of clock readings of new deterministic event. Since we have to consider two clock readings, two possible orderings may occur. For $a_1 \leq a_2$ we get:

$$
\begin{aligned}
p_{ij}(a_1,a_2) &= P\left\{S_{n+1}=s_j, C_{n+1}(l(j))\le a_1, C_{n+1}(m(j))\le a_2 \;\middle|\; S_n = s_i\right\} \\
&= \sum_{z_1\in Z_1}\sum_{z_2\in Z_2} P\left\{s_i \xrightarrow[a_1]{\exp^*} z_1\right\}\cdot P\left\{z_1 \xrightarrow[a_2-a_1]{\exp^*} z_2\right\}\cdot P\left\{z_2 \xrightarrow[D-a_2]{\exp^*} s_j\right\} \\
&\quad + \sum_{z\in Z_2} P\left\{s_i \xrightarrow[a_1]{\exp^*} z\right\}\cdot P\left\{z \xrightarrow[D-a_1]{\exp^*} s_j\right\} \\
&= \sum_{z_1\in Z_1}\sum_{z_2\in Z_2} \tilde{p}_{s_i,z_1}(a_1)\cdot \tilde{p}_{z_1,z_2}(a_2-a_1)\cdot \tilde{p}_{z_2,s_j}(D-a_2) + \sum_{z\in Z_2}\tilde{p}_{s_i,z}(a_1)\cdot\tilde{p}_{z,s_j}(D-a_1)
\end{aligned}
\tag{14}
$$

with $Z_1 = S_{det1}\cap S\left(e_{l(j)}\right)\cap SMC(s_i)\cap SRS\left(s_j\right)$ and $Z_2 = S\left(e_{l(j)}, e_{m(j)}\right)\cap SMC(s_i)\cap SRS\left(s_j\right)$.

In a similar way, for $a_2 \le a_1$, we get:

$$
p_{ij}(a_1,a_2) = \sum_{z_1\in Z_1}\sum_{z_2\in Z_2} \tilde{p}_{s_i,z_1}(a_2)\cdot \tilde{p}_{z_1,z_2}(a_1-a_2)\cdot \tilde{p}_{z_2,s_j}(D-a_1) + \sum_{z\in Z_2}\tilde{p}_{s_i,z}(a_2)\cdot\tilde{p}_{z,s_j}(D-a_2) \tag{15}
$$

with $Z_1 = S\left(e_{m(j)}\right)\cap SMC(s_i)\cap SRS\left(s_j\right)$ and Z_2 as above.

Next, we consider the derivation of kernel elements of the submatrix $\mathbf{P}_{22}(c_1,a_1)$. If the deterministic event scheduled in state s_i at time nD, denoted by $e_{l(i)}$, cannot be canceled, this event occurs with probability one exactly at time $nD+c_1$. Again, two possible orderings of clock readings may occur. For $c_1 \le a_1$, using (11), we get:

$$
\begin{aligned}
p_{ij}(c_1,a_1) &= P\left\{S_{n+1}=s_j, C_{n+1}(l(j))\le a_1 \;\middle|\; S_n = s_i, C_n(l(i)) = c_1\right\} \\
&= \sum_{(z_1,z_1',e_{l(i)})}\sum_{z_2\in Z} P\left\{s_i \xrightarrow[c_1]{\exp^*} z_1\right\}\cdot P\left\{z_1' \xrightarrow[a_1-c_1]{\exp^*} z_2\right\}\cdot P\left\{z_2 \xrightarrow[D-a_1]{\exp^*} s_j\right\} \\
&\quad + \sum_{(z,z',e_{l(i)})} P\left\{s_i \xrightarrow[c_1]{\exp^*} z\right\}\cdot P\left\{z' \xrightarrow[D-c_1]{\exp^*} s_j\right\} \\
&= \sum_{(z_1,z_1',e_{l(i)})}\sum_{z_2\in Z} \tilde{p}_{s_i,z_1}(c_1)\cdot \tilde{p}_{z_1',z_2}(a_1-c_1)\cdot \tilde{p}_{z_2,s_j}(D-a_1) + \sum_{(z,z',e_{l(i)})}\tilde{p}_{s_i,z}(c_1)\cdot\tilde{p}_{z',s_j}(D-c_1)
\end{aligned}
\tag{16}
$$

with $Z = S_{det1}\cap S\left(e_{l(i)}\right)\cap SRS\left(s_j\right)$.

For $a_1 < c_1$, we get:

$$
\begin{aligned}
p_{ij}(c_1,a_1) &= P\left\{S_{n+1}=s_j, C_{n+1}(l(j))\le a_1 \;\middle|\; S_n = s_i, C_n(l(i)) = c_1\right\} \\
&= \sum_{(z_2,z_2',e_{l(j)})}\sum_{z_1\in Z} \tilde{p}_{s_i,z_1}(a_1)\cdot \tilde{p}_{z_1,z_2}(c_1-a_1)\cdot \tilde{p}_{z_2',s_j}(D-c_1)
\end{aligned}
\tag{17}
$$

with $Z = SMC(s_i)\cap S\left(e_{l(i)}, e_{l(j)}\right)$. If the deterministic event $e_{l(i)}$ may be canceled, kernel elements of the submatrix $\mathbf{P}_{22}(c_1,a_1)$ can be derived in a similar way.

Finally, let us consider the derivation of kernel elements of the submatrix $\mathbf{P}_{33}(c_1,c_2,a_1,a_2)$ as the most general case. Recall from Figure 2 that in general 24 orderings of clock readings c_1, c_2, a_1, and a_2 are possible. In Eq. (18) we derive how to compute kernel elements of the form $p_{ij}(c_1,c_2,a_1,a_2)$ for the ordering $c_1 \le c_2 \le a_1 \le a_2$ under the assumption that both deterministic events $e_{l(i)}$ and $e_{m(i)}$ cannot be canceled.

$$
\begin{aligned}
&p_{ij}(c_1, c_2, a_1, a_2) \\
&\quad = P\left\{S_{n+1} = s_j, C_{n+1}(l(j)) \le a_1, C_{n+1}(m(j)) \le a_2 \;\middle|\; S_n = s_i, C_n(l(i)) = c_1, C_n(m(i)) = c_2\right\} \\
&\quad = \sum_{(z_1, z_1', e_{l(i)})} \sum_{(z_2, z_2', e_{m(i)})} \sum_{z_3 \in Z_1} \sum_{z_4 \in Z_2} \tilde{p}_{s_i, z_1}(c_1) \cdot \tilde{p}_{z_1', z_2}(c_2 - c_1) \cdot \tilde{p}_{z_2', z_3}(a_1 - c_2) \\
&\qquad\qquad\qquad\qquad \cdot \tilde{p}_{z_3, z_4}(a_2 - a_1) \cdot \tilde{p}_{z_4, s_j}(D - a_2) \qquad (18) \\
&\qquad + \sum_{(z_1, z_1', e_{l(i)})} \sum_{(z_2, z_2', e_{m(i)})} \sum_{z_3 \in Z_1} \tilde{p}_{s_i, z_1}(c_1) \cdot \tilde{p}_{z_1', z_2}(c_2 - c_1) \cdot \tilde{p}_{z_2', z_3}(a_1 - c_2) \cdot \tilde{p}_{z_3, s_j}(D - a_1) \\
&\qquad + \sum_{(z_1, z_1', e_{l(i)})} \sum_{(z_2, z_2', e_{m(i)})} \sum_{z_3 \in Z_1} \tilde{p}_{s_i, z_1}(c_1) \cdot \tilde{p}_{z_1', z_2}(c_2 - c_1) \cdot \tilde{p}_{z_2', z_3}(a_2 - c_2) \cdot \tilde{p}_{z_3, s_j}(D - a_2) \\
&\qquad + \sum_{(z_1, z_1', e_{l(i)})} \sum_{(z_2, z_2', e_{m(i)})} \tilde{p}_{s_i, z_1}(c_1) \cdot \tilde{p}_{z_1', z_2}(c_2 - c_1) \cdot \tilde{p}_{z_2', z_3}(D - c_2)
\end{aligned}
$$

with appropriately defined sets Z_1 and Z_2.

As shown in (12) to (18), kernel entries can be computed as summation of appropriately selected transient state probabilities of CTMCs. ■

3.3 Detection of Constant Kernel Elements

In this section, we state sufficient conditions on the building blocks of the GSMP under which kernel elements are constant because jump probabilities of the GSSMC are independent of clock readings. This is for example the case in (multiserver) queueing systems if the arrival process is independent from the number of customers in the queue. For arbitrary GSMPs, a necessary condition for kernel elements of the GSSMC to be constant is that clock readings of new deterministic events at time (n+1)D do not depend on the occurrence of exponential events in [nD, D). For GSMPs with at most two deterministic events concurrently active, this implies $p_{ij}(c_1,a_1) = p_{ij}(c_1)$ and $p_{ij}(c_1,c_2,a_1,a_2) = p_{ij}(c_1,c_2)$.

Recall that $z \xrightarrow{e_{l(i)}} z'$ denotes the state transition from state z to z' due to the occurrence of deterministic event $e_{l(i)}$. Recall also that we write $s_i \xrightarrow[t]{\exp^*} s_j$ for a sequence of exponential events from s_i to s_j in time t. Putting it together, we write a path $s_i \xrightarrow[c_1]{\exp^*} z \xrightarrow{e_{l(i)}} z' \xrightarrow[D-c_1]{\exp^*} s_j$ for a sequence of exponential events in (0, c_1] followed by the occurrence of deterministic event $e_{l(i)}$ and another sequence of exponential events in (c_1, D]. If for each such path holds $z' \in S_{det1}$, then the clock reading of the new deterministic event does not depend on the occurrence of exponential events in [nD, (n+1)D).

We derive conditions on the structure of the GSMP, so that the jump probability corresponding to a path $s_i \xrightarrow[c_1]{\exp^*} z \xrightarrow{e_{l(i)}} z' \xrightarrow[D-c_1]{\exp^*} s_j$ leads to the same jump probability as $s_i \xrightarrow[D]{exp^*} z$ with $z' = s_j$. Noting that in the latter case the corresponding transient state probability of the SMC is independent of the clock reading c_1, the kernel element is constant.

Consider two states s_i and $s_j \in S_{det1}$. A prerequisite for the computation of the corresponding kernel element according to (16) constitutes the derivation of all feasible paths $s_i \xrightarrow[c_1]{exp^*} z \xrightarrow{e_{l(i)}} z' \xrightarrow[D-c_1]{exp^*} s_j$ from the reachability graph of the GSMP. We group all such paths into classes such that in each class, paths comprise of the same number of events, say L. That is the deterministic event $e_{l(i)}$ and L-1 occurrences of exponential events ($L \geq 1$). If one of the subpaths $s_i \xrightarrow{exp^*} z$ or $z' \xrightarrow{exp^*} s_j$ contains a cycle, we consider just one round of this cycle. For all path of class m (m = 1,2,..,M), we define a rectangular matrix Γ_m:

$$\Gamma_m \overset{def}{=} \Gamma_m(k,l) = \begin{bmatrix} \tilde{s}_{1,0} & \tilde{s}_{1,1} & \cdots & \tilde{s}_{1,L-1} & \tilde{s}_{1,L} \\ \tilde{s}_{2,0} & \tilde{s}_{2,1} & & & \tilde{s}_{2,L} \\ \vdots & & \ddots & & \vdots \\ \tilde{s}_{K,0} & \tilde{s}_{K,1} & \cdots & \tilde{s}_{K,L-1} & \tilde{s}_{K,L} \end{bmatrix} \quad \begin{array}{l} \text{with } \tilde{s}_{k,0} \overset{def}{=} s_i \text{ and } \tilde{s}_{k,L} \overset{def}{=} s_j \\ \text{for } k = 1,2,...,K \end{array} \qquad (19)$$

Theorem 2 states sufficient conditions on the building blocks of the GSMP under which the *proportionate jump probability*, p_{ijm}, corresponding to class Γ_m is constant. If for some s_i, $s_j \in S_{det1}$ the proportionate jump probabilities of all classes are constant, then the kernel element $p_{ij}(c_1)$ is constant, too, since applying (16) $p_{ij}(c_1) = \sum_m p_{ijm}$.

Theorem 2 (Conditions under which kernel elements are constant)

Let $\mathbf{P}(c_1,c_2,a_1,a_2)$ be the transition kernel of the embedded GSSMC $\{X_n : n \geq 0\}$ and let s_i and $s_j \in S_{det1}$ with the matrix $\Gamma_m = \Gamma_m(k,l)$ representing all paths of class m. Assume that the deterministic event $e_{l(i)}$ cannot be canceled in any path of Γ_m. Then, the corresponding proportionate jump probability p_{ijm} is constant, if the following conditions hold:

(i) The rows of Γ_m can be ordered such that the deterministic event $e_{l(i)}$ occurs at position k (k = 1,2,...,K). That is each path of Γ_m contains the state transition $\tilde{s}_{k,k-1} \xrightarrow{e_{l(i)}} \tilde{s}_{k,k}$. This condition implies that K = L in (19).

(ii) Consider the ordering of (i) for Γ_m, the sets of exponential events scheduled in $\tilde{s}_{k,k-1}$ and in $\tilde{s}_{k,k}$ and corresponding next state probabilities are equal. That is for k = 1,2,..,L this is:

$$E(\tilde{s}_{k,k-1}) \cap E_{exp} = E(\tilde{s}_{k,k}) \cap E_{exp} \text{ and } p(\cdot, \tilde{s}_{k,k-1}, e^*) = p(\cdot, \tilde{s}_{k,k}, e^*)$$

(iii) Consider the ordering of (i) for Γ_m, the multi-set of exponential events scheduled and occurring in the k-th row of Γ_m up to position k (and after position k) must be a subset (or superset, respectively) of the corresponding multi-set in the (k+1)-th row. Furthermore, corresponding next state probabilities must be equal. Formally, for k = 1,2,..., L-1 this is:

$$\left\{E(\tilde{s}_{k,0})\cap E_{\exp},\cdots,E(\tilde{s}_{k,k-1})\cap E_{\exp}\right\}\subset\left\{E(\tilde{s}_{k+1,0})\cap E_{\exp},\cdots,E(\tilde{s}_{k+1,k})\cap E_{\exp}\right\}$$
$$\text{and}\quad \left\{e^*_{k,0},\cdots,e^*_{k,k-2}\right\}\subset\left\{e^*_{k+1,0},\cdots,e^*_{k+1,k-1}\right\}$$
$$\left\{E(\tilde{s}_{k,k})\cap E_{\exp},\cdots,E(\tilde{s}_{k,L})\cap E_{\exp}\right\}\supset\left\{E(\tilde{s}_{k+1,k+1})\cap E_{\exp},\cdots,E(\tilde{s}_{k+1,L})\cap E_{\exp}\right\}$$
$$\text{and}\quad \left\{e^*_{k,k},\cdots,e^*_{k,L}\right\}\supset\left\{e^*_{k+1,k+1},\cdots,e^*_{k+1,L}\right\}$$

Note that condition (ii) implies that for each path of Γ_m the states $\tilde{s}_{k,k-1}$ and $\tilde{s}_{k,k}$ can be combined yielding one path of exponential events $s_i \xrightarrow[D]{\exp^*} s_j$ as illustrated in Figure 3.

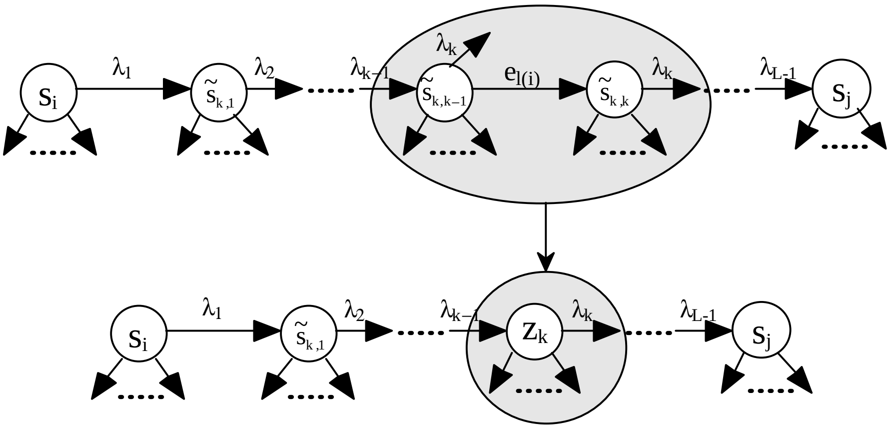

Figure 3. Combining states in SMCs leading to constant jump probabilities

Proof: We have to show that the proportionate jump probability $p_{ijm}(c_1)$ is independent of the clock reading c_1. Assuming the paths of Γ_m are ordered according to condition (i). Then, using (16) the proportionate jump probability p_{ijm} is given by:

$$p_{ijm}(c_1)=\sum_{k=1}^{L}P\left\{s_i \xrightarrow[c_1]{\exp^*} \tilde{s}_{k,k-1}\right\}\cdot P\left\{\tilde{s}_{k,k} \xrightarrow[D-c_1]{\exp^*} s_j\right\}=\sum_{k=1}^{L}\tilde{p}_{s_i,\tilde{s}_{k,k-1}}(c_1)\cdot\tilde{p}_{\tilde{s}_{k,k},s_j}(D-c_1) \qquad (20)$$

Because of condition (ii), the same set of exponential events is scheduled in states $\tilde{s}_{k,k-1}$ and $\tilde{s}_{k,k}$ and corresponding next state probabilities are equal for k = 1,2,...,L. Thus, we can combine the two states to one state z_k as illustrated in Figure 3. Subsequently, we can rewrite (20) as:

$$p_{ijm}(c_1)=\sum_{k=1}^{L}\tilde{p}_{s_i,z_k}(c_1)\cdot\tilde{p}_{z_k,s_j}(D-c_1) \qquad (21)$$

Recall that according to condition (iii), each path contains the same (multi)-set of L-1 scheduled and occurring exponential events and corresponding next state probabilities are equal. Note that Eq. (21) can be interpreted as the convolution of two exponential phase-type random variables given by the exponential events occurring in (0, c_1] and in (c_1, D]. Since according to (i) in Γ_m the deterministic event $e_{l(i)}$ occurs exactly once at position k, the summation in Eq. (21) is taken over all possible cases. Thus, the jump probability p_{ijm} is given by the probability that the considered L-1 exponential events occur in (0,D]. This immediately leads to $p_{ijm}(c_1)=p_{ijm}=\tilde{p}_{s_i,\tilde{s}_{L,L-1}}(D)$. ■

As already mentioned, kernel elements of the form $p_{ij}(c_1,c_2,a_1,a_2) = p_{ij}(c_1,c_2)$ may also be constant. In fact, assuming $c_1 \le c_2$ kernel elements $p_{ij}(c_1,c_2,a_1,a_2)$ are constant, if the summation according to (18) of all paths of the form $s_i \xrightarrow[c_1]{exp^*} z_1 \xrightarrow{e_{l(i)}} z_1' \xrightarrow[c_2-c_1]{exp^*} z_2 \xrightarrow{e_{m(i)}} z_2' \xrightarrow[D-c_2]{exp^*} s_j$ leads to the same jump probability as $s_i \xrightarrow[D]{exp^*} z_1$ with $z_1' = z_2$ and $z_2' = s_j$. As above, we group all feasible paths of this form into classes of equal length and define for each class a rectangular matrix according to (19). Due to space limitations, we just state informally conditions ensuring that a proportionate jump probability p_{ijm} is independent of the clock readings c_1 and c_2. Assume that the deterministic events $e_{l(i)}$ and $e_{m(i)}$ cannot be canceled in any path of Γ_m. Subsequently, the proportionate jump probability of class m, p_{ijm}, is constant, if the following conditions hold:

(iv) The rows of Γ_m can be ordered such that the deterministic events $e_{l(i)}$ and $e_{m(i)}$ occur at positions k and l with k = 1,2,...,L-1 and l = k+1,...,L. This condition implies that $K = \frac{1}{2}L(L-1)$ in (19).

(v) The (multi)-set of L-2 exponential events which occur is equal in each path of Γ_m.

(vi) The sets of exponential events scheduled in states $\tilde{s}_{k,l}$ in which a deterministic events occurs and the next state $\tilde{s}_{k,l}'$ are the same and corresponding next state probabilities are equal.

3.4 Detection of Symmetric State Probabilities

When in the GSSMC X_n clock readings of two concurrent deterministic events $e_{l(j)}$ and $e_{m(j)}$ are always set in an equal way, the time-dependent and stationary probabilities $\pi_j^{(n)}(a_1,a_2)$ and $\pi_j(a_1,a_2)$ are symmetric functions with respect to a_1 and a_2. This implies that for $s_j \in S_{det2}$ with $E(s_j) \cap E_{det} = \{e_{l(j)}, e_{m(j)}\}$ for each feasible path of the form $s_i \xrightarrow{exp^*} z_1 \xrightarrow{exp^*} s_j$ with $E(z_1) \cap E_{det} = \{e_{l(j)}\}$ exits a corresponding path $s_i \xrightarrow{exp^*} \hat{z}_1 \xrightarrow{exp^*} s_j$ with $E(\hat{z}_1) \cap E_{det} = \{e_{m(j)}\}$ such that $\pi_j(a_1,a_2) = \pi_j(a_2,a_1)$. Similar conditions can be derived for other forms of feasible paths to state $s_j \in S_{det2}$. Recall that we write $s_i \xrightarrow{exp^*} s_j$ For a possibly empty sequence of exponential events from s_i to s_j, For a state transition from s_i to s_j due to the occurrence of exactly one exponential event, we write $s_i \xrightarrow{exp} s_j$. In the most general setting, these conditions are met when the reachability graph of the GSMP is isomorphic to the reachability graph of a GSMP in which all arcs labeled with $e_{l(j)}$ are replaced by arcs labeled $e_{m(j)}$ and vice-versa. However, in general checking graph isomorphism may require even for weighted directed graphs (i.e., reachability graphs of GSMPs) a high computational effort. Theorem 3 states sufficient condition on the building blocks of the GSMP that just depend on state transitions to/from immediate neighbor states in

Theorem 3 (Conditions under which state probabilities are symmetric)

Let $\{X(t): t \geq 0\}$ be a finite state space GSMP with exponential and deterministic events. Consider a state $s_j \in S_{det2}$ in which the both deterministic events $e_{l(i)}$ and $e_{m(i)}$ are concurrently active. Let z_1, $z_2 \in S_{det1}$ with $E(z_1) \cap E_{det} = \{e_{l(i)}\}$ and $E(z_2) \cap E_{det} = \{e_{m(i)}\}$. Then, $\pi_j(a_1, a_2) = \pi_j(a_2, a_1)$, if the following conditions are met:

(i) For each $s_i \in S_{exp}$ with transition $s_i \xrightarrow{exp} z_1$, exists a transition $s_i \xrightarrow{exp} z_2$ which comprises of the same exponential event e* and equal next state probability $p(\cdot, s_i, e^*)$.

(ii) For z_1, $z_2 \in S_{det1}$ as defined above with transition $z_1 \xrightarrow{exp} s_j$, exists a transition $z_2 \xrightarrow{exp} s_j$ which comprises of the same exponential event e* and equal next state probabilities; $p(\cdot, z_1, e^*) = p(\cdot, z_2, e^*)$. Furthermore, for each transition $z_1 \xrightarrow{e_{l(j)}} z'$ exists a corresponding transition $z_2 \xrightarrow{e_{m(j)}} z'$ with $z' \in S_{det1}$.

(iii) For each $s_i \in S_{det2}$ with $E(s_i) \cap E_{det} = \{e_{l(j)}, e_{m(j)}\}$ and transitions $s_i \xrightarrow{e_{l(j)}} z_1'$ and $s_i \xrightarrow{e_{m(j)}} z_2'$ holds either $E(z_1') - \{e_{m(j)}\} = E(z_2') - \{e_{l(j)}\}$ with $z_1', z_2' \in S_{det1}$ or $z_1' = z_2'$ with $z_1', z_2' \in S_{det2}$. That former condition says that the exponential events scheduled in the new state reached by the occurrence of one of the deterministic events and corresponding next state probabilities are the same.

the reachability graph. Thus, these conditions can be checked easily. Note, these conditions apply to all multiserver queueing systems (e.g., MAP/D/2/K). However, there exists GSMPs of the considered class in which state probabilities are symmetric, even if the conditions of Theorem 3 are not met. Due to space limitations, we omit the proof of this theorem.

4 Impact of Theorems for the Efficient Numerical Analysis

4.1 The System of Integral Equations

In order to illustrate the impact of results presented in the previous section, we recall the systems of time-dependent and stationary equations of the GSSMC as introduced in [9], [10]. To write these systems of Fredholm integral equations in vector notation, we define three vectors of state probabilities for the states of S_{exp}, S_{det1}, and, S_{det2} respectively.

$$\begin{aligned}
&\pi_{exp}^{(n)} = \left(\pi_1^{(n)}, \pi_2^{(n)}, \ldots, \pi_{N_1}^{(n)}\right) \\
&\pi_{det1}^{(n)}(a_1) = \left(\pi_{N_1+1}^{(n)}(a_1), \pi_{N_1+2}^{(n)}(a_1), \ldots, \pi_{N_1+N_2}^{(n)}(a_1)\right) \\
&\pi_{det2}^{(n)}(a_1, a_2) = \left(\pi_{N_1+N_2+1}^{(n)}(a_1, a_2), \pi_{N_1+N_2+2}^{(n)}(a_1, a_2), \ldots, \pi_N^{(n)}(a_1, a_2)\right)
\end{aligned} \tag{22}$$

To further simplify the notation in the systems of integral equation (17) to (19), we introduce two vectors $y^{(n)}(c_1)$ and $z^{(n)}(c_1,c_2)$ for the derivatives of state probabilities as:

$$y^{(n)}(c_1) \overset{\text{def}}{=} \frac{d\pi^{(n)}_{\det 1}(c_1)}{dc_1} \quad \text{and} \quad z^{(n)}(c_1,c_2) \overset{\text{def}}{=} \frac{\partial^2 \pi^{(n)}_{\det 2}(c_1,c_2)}{\partial c_1 \partial c_2} \tag{23}$$

As shown in [9], the GSSMC allows the numerical analysis of GSMPs with different values D_m for clock settings of deterministic events. However, for ease of exposition, we recall just the systems of time-dependent and stationary equations of the GSSMC under the restriction that all deterministic events have the same delay D and that concurrent deterministic events cannot be canceled. Then, using the submatrices $\mathbf{P}_{ij}(.)$ of the transition kernel defined in (7) together with (22) and (23), time-dependent state probabilities for the GSMP at instants of time nD are given by [10]:

$$\pi^{(n+1)}_{\exp} = \pi^{(n)}_{\exp} \cdot \mathbf{P}_{11} + \int_0^D y^{(n)}(c_1) \cdot \mathbf{P}_{21}(c_1) dc + \int_0^D \int_0^{c_2} z^{(n)}(c_1,c_2) \cdot \mathbf{P}_{31}(c_1,c_2) + z^{(n)}(c_2,c_1) \cdot \mathbf{P}_{31}(c_2,c_1) dc_1 dc_2 \tag{24}$$

$$\begin{aligned} \pi^{(n+1)}_{\det 1}(a_1) = {} & \pi^{(n)}_{\exp} \cdot \mathbf{P}_{12}(a_1) + \int_0^{a_1} y^{(n)}(c_1) \cdot \mathbf{P}_{22}(c_1,a_1) dc + \int_{a_1}^{D} y^{(n)}(c_1) \cdot \mathbf{P}_{22}(c_1,a_1) dc_1 \\ & + \int_0^{a_1} \int_0^{c_2} z^{(n)}(c_1,c_2) \cdot \mathbf{P}_{32}(c_1,c_2,a_1) + z^{(n)}(c_2,c_1) \cdot \mathbf{P}_{32}(c_2,c_1,a_1) dc_1 dc_2 \\ & + \int_{a_1}^{D} \int_0^{a_1} z^{(n)}(c_1,c_2) \cdot \mathbf{P}_{32}(c_1,c_2,a_1) + z^{(n)}(c_2,c_1) \cdot \mathbf{P}_{32}(c_2,c_1,a_1) dc_1 dc_2 \end{aligned} \tag{25}$$

$$\begin{aligned} \pi^{(n+1)}_{\det 2}(a_1,a_2) = {} & \pi^{(n)}_{\exp} \cdot \mathbf{P}_{13}(a_1,a_2) + \int_0^{a_1} y^{(n)}(c) \cdot \mathbf{P}_{23}(c_1,a_1,a_2) dc_1 + \int_{a_1}^{a_2} y^{(n)}(c_1) \cdot \mathbf{P}_{23}(c_1,a_1,a_2) dc_1 \\ & + \int_0^{a_1} \int_0^{c_2} z^{(n)}(c_1,c_2) \cdot \mathbf{P}_{33}(c_1,c_2,a_1,a_2) + z^{(n)}(c_2,c_1) \cdot \mathbf{P}_{33}(c_2,c_1,a_1,a_2) dc_1 dc_2 \\ & + \int_{a_1}^{a_2} \int_0^{a_1} z^{(n)}(c_1,c_2) \cdot \mathbf{P}_{33}(c_1,c_2,a_1,a_2) + z^{(n)}(c_2,c_1) \cdot \mathbf{P}_{33}(c_2,c_1,a_1,a_2) dc_1 dc_2 \\ & \text{for } a_1 \le a_2 \end{aligned} \tag{26}$$

$$\begin{aligned} \pi^{(n+1)}_{\det 2}(a_1,a_2) = {} & \pi^{(n)}_{\exp} \cdot \mathbf{P}_{13}(a_1,a_2) + \int_0^{a_2} y^{(n)}(c_1) \cdot \mathbf{P}_{23}(c_1,a_1,a_2) dc_1 + \int_{a_2}^{a_1} y^{(n)}(c_1) \cdot \mathbf{P}_{23}(c_1,a_1,a_2) dc_1 \\ & + \int_0^{a_2} \int_0^{c_2} z^{(n)}(c_1,c_2) \cdot \mathbf{P}_{33}(c_1,c_2,a_1,a_2) + z^{(n)}(c_2,c_1) \cdot \mathbf{P}_{33}(c_2,c_1,a_1,a_2) dc_1 dc_2 \\ & + \int_{a_2}^{a_1} \int_0^{a_2} z^{(n)}(c_1,c_2) \cdot \mathbf{P}_{33}(c_1,c_2,a_1,a_2) + z^{(n)}(c_2,c_1) \cdot \mathbf{P}_{33}(c_2,c_1,a_1,a_2) dc_1 dc_2 \end{aligned} \tag{27}$$

for $a_2 \le a_1$

where $0 \le a_1, a_2 \le D$ and $\pi_{det1}(0) = \pi_{det2}(0, c_2) = \pi_{det2}(c_1, 0) = 0$.

Taking the limits $n \to \infty$ in (24) to (27) and using some algebra, we derive a system of Fredholm integral equations in:

$$0 = \pi_{exp} \cdot (\mathbf{P}_{11} - \mathbf{I}) + \int_0^D y(c_1) \cdot \mathbf{P}_{21}(c_1) dc_1 + \int_0^D \int_0^{c_2} z(c_1, c_2) \cdot \mathbf{P}_{31}(c_1, c_2) + z(c_2, c_1) \cdot \mathbf{P}_{31}(c_2, c_1) dc_1 dc_2 \quad (28)$$

$$\begin{aligned} 0 = \pi_{exp} \cdot \mathbf{P}_{12}(a_1) &+ \int_0^{a_1} y(c_1) \cdot (\mathbf{P}_{22}(c_1, a_1) - \mathbf{I}) dc_1 + \int_{a_1}^D y(c_1) \cdot \mathbf{P}_{22}(c_1, a_1) dc_1 \\ &+ \int_0^{a_1} \int_0^{c_2} z(c_1, c_2) \cdot \mathbf{P}_{32}(c_1, c_2, a_1) + z(c_2, c_1) \cdot \mathbf{P}_{32}(c_2, c_1, a_1) dc_1 dc_2 \\ &+ \int_{a_1}^D \int_0^{a_1} z(c_1, c_2) \cdot \mathbf{P}_{32}(c_1, c_2, a_1) + z(c_2, c_1) \cdot \mathbf{P}_{32}(c_2, c_1, a_1) dc_1 dc_2 \end{aligned} \quad (29)$$

$$\begin{aligned} 0 = \pi_{exp} \cdot \mathbf{P}_{13}(a_1, a_2) &+ \int_0^{a_1} y(c) \cdot \mathbf{P}_{23}(c_1, a_1, a_2) dc + \int_{a_1}^{a_2} y(c) \cdot \mathbf{P}_{23}(c_1, a_1, a_2) dc \\ &+ \int_0^{a_1} \int_0^{c_2} z(c_1, c_2) \cdot (\mathbf{P}_{33}(c_1, c_2, a_1, a_2) - \mathbf{I}) + z(c_2, c_1) \cdot (\mathbf{P}_{33}(c_2, c_1, a_1, a_2) - \mathbf{I}) dc_1 dc_2 \\ &+ \int_{a_1}^{a_2} \int_0^{a_1} z(c_1, c_2) \cdot (\mathbf{P}_{33}(c_1, c_2, a_1, a_2) - \mathbf{I}) + z(c_2, c_1) \cdot \mathbf{P}_{33}(c_2, c_1, a_1, a_2) dc_1 dc_2 \end{aligned} \quad (30)$$

for $a_1 \le a_2$

$$\begin{aligned} 0 = \pi_{exp} \cdot \mathbf{P}_{13}(a_1, a_2) &+ \int_0^{a_2} y(c) \cdot \mathbf{P}_{23}(c, a_1, a_2) dc + \int_{a_2}^{a_1} y(c) \cdot \mathbf{P}_{23}(c, a_1, a_2) dc \\ &+ \int_0^{a_2} \int_0^{c_2} z(c_1, c_2) \cdot (\mathbf{P}_{33}(c_1, c_2, a_1, a_2) - \mathbf{I}) + z(c_2, c_1) \cdot (\mathbf{P}_{33}(c_2, c_1, a_1, a_2) - \mathbf{I}) dc_1 dc_2 \\ &+ \int_{a_2}^{a_1} \int_0^{a_2} z(c_1, c_2) \cdot \mathbf{P}_{33}(c_1, c_2, a_1, a_2) + z(c_2, c_1) \cdot (\mathbf{P}_{33}(c_2, c_1, a_1, a_2) - \mathbf{I}) dc_1 dc_2 \end{aligned} \quad (31)$$

for $a_2 \le a_1$

where $0 \le a_1, a_2 \le D$ and $\pi_{det1}(0) = \pi_{det2}(0, c_2) = \pi_{det2}(c_1, 0) = 0$. Having solved (28) to (31) for π_{exp}, $y^{(n)}(c_1)$ and $z^{(n)}(c_1, c_2)$, the stationary or time-averaged state probabilities $\pi_i(D)$, and $\pi_i(D, D)$ of the GSSMC are derived by numerical integration.

Applying Theorem 3, we can detect when state probabilities are symmetric with respect to clocks of concurrent deterministic events. This leads to $\pi_i(a_1, a_2) = \pi_i(a_2, a_1)$ for $0 < a_1, a_2 \le D$. As a consequence, Eqs. (27) and (31) of the systems of integral equations (24) to (27) and (28) to (31) can be omitted in the numerical analysis of the corresponding GSMP.

4.2 Easy Numerical Computation of the Transition Kernel

The result that all elements of the transition kernel of the GSSMC can be expressed by summation of transient state probabilities of CTMCs (Theorem 1) reduces the computation of jump probabilities of a stochastic process with continuous state space, i.e., a GSSMC to transient analysis of a number of simple stochastic processes, i.e., the SMCs. An efficient numerical method for transient analysis of CTMCs is the randomization technique [7]. Note that the computation of the transient state probability vector $\pi_i(D)$ of $SMC(s_i)$ by randomization also yields transient state probability vectors $\pi_i(t)$ for $0 < t < D$ as intermediate results. As a consequence, the numerical computation of the transition kernel of the GSSMC requires asymptotically the same effort as the numerical computation of the probability matrix **P** of the discrete-time Markov chain embedded in the Markov regenerative process underlying discrete-event systems without concurrent deterministic events [1], [3], [8].

To illustrate the concept of SMCs, we consider a finite-capacity multiserver queueing systems. The system comprises of two identical servers with constant service time D and one queue with limited capacity K. Customers arrive according to Poisson distributions (λ_1, λ_2, .., λ_N) whose parameter is controlled by an N-state CTMC with birth-death structure, i.e, a Markov modulated Poisson process. When an arriving customer finds an empty system, it enters server 1 with probability p and server 2 with (1-p). This queueing system is known as the MMPP/D/2/K queue. The state of the corresponding GSMP is determined by the number of customers in the system and by the state of the arrival process. When just a single customer resides in the system, we distinguish whether this customer is served at server 1 or server 2. The number of states of the GSSMC underlying the MMPP/D/2/K is given by $N(K+2)$. In N states are only exponential events active whereas in 2N states exactly one deterministic event is active. The number of states in which two deterministic events are concurrently active is given by $N(K-1)$. Assuming N = 2, Figure 4 shows the reachability graph of the GSMP

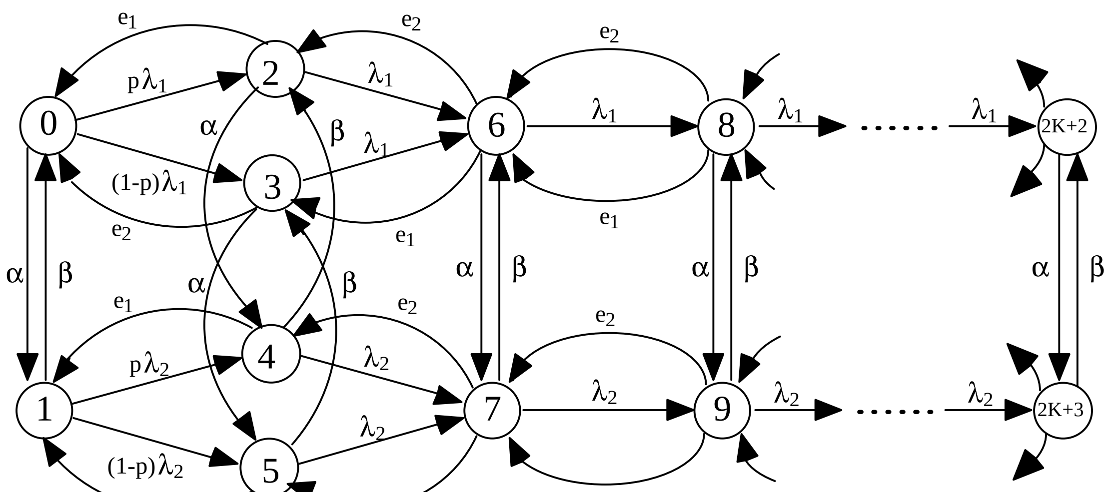

Figure 4. Reachability graph of the GSMP underlying the MMPP/D/2/K queue

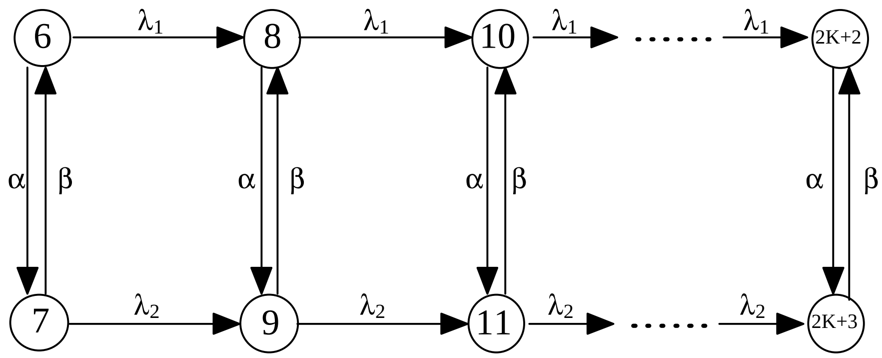

Figure 5. Subordinated Markov chain of states 6 and 7 of the MMPP/D/2/K queue

underlying the MMPP/D/2/K queue. The deterministic service is represented by arcs labeled with events e_1 and e_2. State changes due to exponential events are labeled with their rate parameters α, β, λ_1, and λ_2. Figure 5 shows the SMCs of states 6 and 7. Applying Theorem 3, we detect that for p = ½ holds $\pi_i(a_1, a_2) = \pi_i(a_2, a_1)$ for all states $s_i \in S_{det2}$.

4.3 Exploitation of Constant Kernel Elements

The detection of constant kernel elements (Theorem 2) implies that corresponding integral expressions in the systems of Fredholm equations (24) to (27) and (28) to (31) vanish. Table 1 shows the number of nonzero entries of the transition kernel of the GSSMC underlying the MMPP/D/2/K queue for increasing model size; i.e., K = 1000 to 10000 and provides percentages for each of the five different types of kernel elements. Note that this table shows the number of kernel elements whose analytic expressions are nonzero. The employment of dynamic sparsing of kernel elements in the practical computational scheme leads to a substantial reduction of nonzero elements and, thus, of memory requirements. From Table 1 we observed that for this class of GSMP, i.e., GSMPs corresponding to multiserver queueing

States of GSSMC	Nonzero entries	Constant entries	Functionals in 1 variable	Functionals in 2 variables	Functionals in 3 variables	Functionals in 4 variables
2004	2004997	99,30 %	0,20 %	0,49 %	$2{,}5\cdot10^{-4}$ %	$1{,}0\cdot10^{-4}$ %
4008	8009997	99,65 %	0,10 %	0,25 %	$6{,}3\cdot10^{-5}$ %	$2{,}5\cdot10^{-5}$ %
6012	18014997	99,77 %	0,07 %	0,17 %	$2{,}8\cdot10^{-5}$ %	$1{,}1\cdot10^{-5}$ %
8016	32019997	99,83 %	0,05 %	0,12 %	$1{,}6\cdot10^{-5}$ %	$6{,}3\cdot10^{-6}$ %
10020	50024997	99,86 %	0,04 %	0,10 %	$1{,}0\cdot10^{-5}$ %	$4{,}0\cdot10^{-6}$ %
12024	72029997	99,88 %	0,03 %	0,08 %	$6{,}9\cdot10^{-6}$ %	$2{,}8\cdot10^{-6}$ %
14028	98034997	99,90 %	0,03 %	0,07 %	$5{,}1\cdot10^{-6}$ %	$2{,}0\cdot10^{-6}$ %
16032	128039997	99,91 %	0,02 %	0,06 %	$3{,}9\cdot10^{-6}$ %	$1{,}7\cdot10^{-6}$ %
18036	162044997	99,92 %	0,02 %	0,06 %	$3{,}1\cdot10^{-6}$ %	$1{,}2\cdot10^{-6}$ %
20040	200049997	99,93 %	0,02 %	0,05 %	$2{,}5\cdot10^{-6}$ %	$1{,}0\cdot10^{-6}$ %

Table 1. Classification of elements of the transition kernel of MMPP/D/2/K

systems, more than 99% of nonzero kernel elements are constant. As shown in [10], for quite complex GSMPs (i.e., MMPP/D/2/K queue with K = 10,000 for mission time T = 100) the solution of the system of time-dependent equations requires on a modern workstation about 26 minutes of CPU time, the solution of the corresponding system of stationary equations requires less than 5 minutes of CPU time. Thus, for GSMPs underlying finite-capacity multiserver queueing systems with deterministic service, the exploitation of constant kernel elements in the system of integral equations is key for their highly efficient transient and steady-state analysis.

Conclusions

This paper presented methodological results that provide the foundation for the cost-effective algorithmic generation of the transition kernel of the general state space Markov chain (GSSMC) underlying a GSMP with exponential and deterministic events. Key contributions constitute the formal proof that kernel elements can always be computed by appropriate summation of transient state probabilities of continuous-time Markov chains (Theorem 1). Thus, the computation of the transition kernel of the GSSMC requires asymptotically the same effort as the computation of the probability matrix of the discrete-time Markov chain embedded in the Markov regenerative process underlying discrete-event system without concurrent deterministic events. Furthermore, we derived conditions on the building blocks of the GSMP under which kernel elements are constant; i.e., are independent of clock readings (Theorem 2). We also derive conditions on the building blocks of GSMPs for which state probabilities $\pi_i(a_1, a_2)$ are symmetric with respect to clock readings of deterministic events concurrently active (Theorem 3).

We would like to point out that these properties are valid for GSSMCs with arbitrarily many (i.e., also more than two) deterministic events concurrently active. The exploitation of these properties of the GSSMC considerably reduces the computing time and memory requirements for the numerical solution of the systems of Fredholm integral equations which constitute the systems of time-dependent and stationary equations of the considered class of GSMPs [9], [10]. As a consequence, the implementation of an algorithmic kernel generation based on the presented results together with the already implemented solvers introduced in [9], [10] allow the numerical analysis of complex discrete-event stochastic systems with concurrent deterministic events.

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