

# No Way Out

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# The Timeless Trap

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## 1 Introduction

In the last Petri Net Newsletters [2] we have given an example showing an undesired situation in analysing the Markovian process described by a Generalized Stochastic Petri Net (GSPN;[1]). Another problem is, that it is not always possible to extract the description of a Markov process from a GSPN. This is our topic here.

## 2 The timeless trap

In figure 1 we see a simple producer/consumer-system. Imagine that producing and consuming are time consuming actions, whereas the time to transport the goods can be neglected.

If the consumption is a very fast process the consumer often wastes his time, waiting for the producer to fill the buffer. So the system becomes more efficient if we modify it in the following way. During waiting for an article to arrive in the buffer, the consumer can proceed doing some local tasks. On the other hand we don't want the goods in the buffer to get mouldy. So the consumer should be able to consume articles arriving in the buffer very quickly. Therefore the delay for performing the local tasks should be neglectible. A possible GSPN representing our system is given in figure 2. Now the stochastic process can proceed as follows. After the producer has filled the buffer with one item, the consumer empties the buffer and consumes the article, so that the situation in figure 3 is observable. Because the firing of immediate transitions has priority on that of timed transitions, the

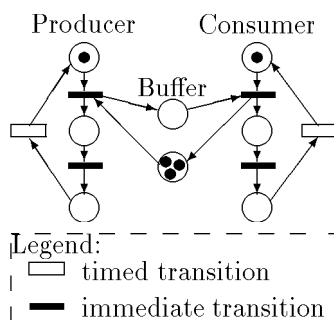


Figure 1: Producer/Consumer-System with limited buffer capacity

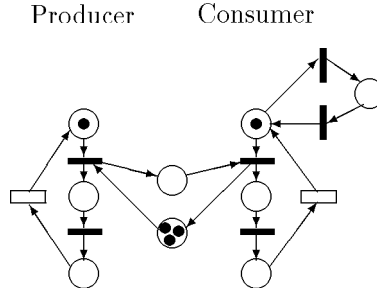


Figure 2: Modified Producer/Consumer-System

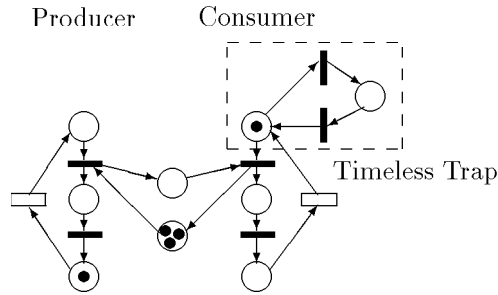


Figure 3: Example of a timeless trap

stochastic process described by this GSPN has to fire the transitions in the "Timeless Trap"-box everlasting.

So how does the Markovian process looks like?

### 3 GSPNs and Markov processes

The reachability set of a GSPN consists of two types of markings (states; see [1]):

- markings enabling only timed transitions are called *tangible states* and
- markings enabling immediate transitions are called *vanishing states*.

A GSPN describes a stochastic process, which dwells a certain (exponentially distributed) amount of time in tangible states and leaves vanishing states immediately. **We assume that the GSPN is bounded, implying a finite reachability set (state space)**. Let  $\lambda_i$  be the firing rate of a timed transition  $t_i$  and  $E(z)$  the set of enabled transitions in marking (state)  $z$ .<sup>1</sup> If  $E(z)$  consists of timed transitions, the probability for  $t_i \in E(z)$  to fire is  $\frac{\lambda_i}{\sum_{t_k \in E(z)} \lambda_k}$ . Otherwise, if  $E(z)$  consists of immediate transitions this probability is specified by the random switch. Looking at the embedded Markov chain, we have the following transition probability matrix:

$$P := A + B := \begin{pmatrix} C & D \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ E & F \end{pmatrix}$$

where

$$c_{ij} = P[i \longrightarrow j; i \in V, j \in V], \quad d_{ij} = P[i \longrightarrow j; i \in V, j \in T],$$

<sup>1</sup>Note that either  $E(z) \subset T_1$  (set of timed transitions) or  $E(z) \subset T_2$  (set of immediate transitions).

$$e_{ij} = P[i \longrightarrow j; i \in T, j \in V], f_{ij} = P[i \longrightarrow j; i \in T, j \in T],$$

$T$  is the set of tangible states,  $V$  is the set of vanishing states,  $T \cap V = \emptyset$ .

The elements of matrix  $A$  are specified by the random switch and the elements of matrix  $B$  can be calculated using the firing rates of timed transitions. Because the sojourn time in vanishing states is zero, we are only interested in tangible states. Before calculating the steady state distribution one has to eliminate all vanishing states yielding a reduced embedded Markov chain with probability transition matrix  $P'$ :

$$P' := \left( p'_{ij} \right), p'_{ij} := f_{ij} + \sum_{r \in V} e_{ir} P[r \longrightarrow j],$$

where  $P[r \longrightarrow j] := P[\text{the stochastic process starts in state } r \text{ and reaches the tangible state } j, \text{ and all states reached in the meantime are vanishing states}], r \in V, j \in T$ .

Let  $P[r \xrightarrow{h} j] := P[\text{the stochastic process starts in state } r \text{ and reaches the tangible state } j \text{ after exactly } h \text{ steps, and all states reached in the meantime are vanishing states}], r \in V, j \in T$ .

$$P[r \longrightarrow j] = \sum_{h=0}^{\infty} P[r \xrightarrow{h} j].$$

Defining

$$G(h) := \left( g_{rj}(h) \right), \text{ where } g_{rj}(h) := P[r \xrightarrow{h} j].$$

yields  $G(h) = C^h \times D$ . With  $G := \left( g_{rj} \right)$ , where  $g_{rj} := P[r \longrightarrow j]$  we have

$$G = \sum_{h=0}^{\infty} G(h) = \sum_{h=0}^{\infty} C^h \times D, \text{ if } \sum_{h=0}^{\infty} C^h \text{ exists.}$$

So  $P' = F + E \times G$  and the steady state distribution  $\tilde{\pi}$  of the reduced embedded Markov chain is given by

$$\tilde{\pi} = \tilde{\pi} \times P'; \sum_{s \in T} \tilde{\pi}_s = 1,$$

if a unique solution for  $\tilde{\pi}$  exists!<sup>2</sup> The steady state distribution  $\pi$  of the Markovian process is given by

$$\pi_j = \frac{\tilde{\pi}_j \times \left( \sum_{k \in E(j)} \lambda_k \right)^{-1}}{\sum_{s \in T} \tilde{\pi}_s \times \left( \sum_{k \in E(s)} \lambda_k \right)^{-1}}, j \in T.$$

A prerequisite for the existence of  $P'$  is the existence of  $\sum_{h=0}^{\infty} C^h$ . How can we characterize the existence of this limit?

Define  $u_i := 1 - \sum_{j=1}^{|V|} c_{ij}$ .

$u_i > 0$  if the sum in line  $i$  is less than 1 and  $u_i = 0$  if it is 1.

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<sup>2</sup>cf. with the negative example in [2]

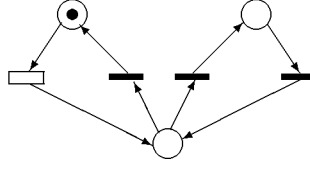


Figure 4: GSPN not satisfying condition NoTT

**Definition 1 (trap; cf. [5])**  $J_0 := \{i \mid u_i = 0\}$  and  $J_1 := \{i \mid u_i > 0\}$ <sup>3</sup>.  
*C has no trap*  $\iff \forall i \in J_0 : \exists j \in J_1$  reachable from  $i$ . Otherwise  $C$  has a trap.

**Theorem 1 ([5])**  $C$  has no trap  $\iff (I - C)^{-1} = \sum_{h=0}^{\infty} C^h$  exists.

Because  $C$  consists of the transition probabilities between vanishing states, we speak of a timeless trap. If there is no timeless trap, we can extract the description of a Markovian process from a GSPN. So how can we avoid such traps?

## 4 Avoidance of timeless traps

**Definition 2 (Condition NoTT; [3])**

*Condition NoTT*<sup>4</sup>  $\iff \forall \tilde{T} \subseteq T_2 : \tilde{T} \neq \emptyset \implies \bullet\tilde{T} \neq \tilde{T}\bullet$ <sup>6</sup>.

**Theorem 2 ([3])** *Condition NoTT*  $\implies$  GSPN has no timeless trap.

Sketch of Proof: Assume there is a timeless trap and let  $\tilde{T}$  be the set of immediate transitions firing everlasting. Define  $\tilde{S} = (\bullet\tilde{T} \setminus \tilde{T}\bullet) \cup (\tilde{T}\bullet \setminus \bullet\tilde{T})$ . If  $\tilde{T} \neq \emptyset$  we have  $\tilde{S} \neq \emptyset$  because otherwise  $\bullet\tilde{T} = \tilde{T}\bullet$  contradicting condition NoTT. There are two cases to consider

a)  $\exists s \in \bullet\tilde{T} \setminus \tilde{T}\bullet$ .

Because of  $s \notin \tilde{T}\bullet$  not all transitions of

$\tilde{T}$  can fire everlasting, contradicting our assumption.

b)  $\exists s \in \tilde{T}\bullet \setminus \bullet\tilde{T}$ .

Because of  $s \notin \bullet\tilde{T}$  and all  $t \in \tilde{T}$  firing everlasting the

GSPN is not bounded, contradicting our global assumption.  $\square$

Condition NoTT is not necessary for the avoidance of timeless traps, which is shown by the GSPN in figure 4. A closer look at the proof of theorem 2 shows that the following statement is valid:

**Theorem 3** *Condition NoTT*  $\implies \exists h_0 \in \mathbb{N} : C^h = 0, \forall h \geq h_0$ .

This property is especially important, if one wants to eliminate vanishing states during state space generation (see e.g. [4]).

Condition NoTT can be verified very easily by the algorithm in figure 5.

<sup>3</sup>If the stochastic process is in a state  $i \in J_0$ , it can only transit to another vanishing state, being in state  $j \in J_1$  there is a positive probability to reach a tangible state.

<sup>4</sup>NoTT stands for No Timeless Trap.

<sup>5</sup> $T_2$  is the set of immediate transitions (see [2])

<sup>6</sup>as usual  $\bullet t$  denotes the set of input places and  $t\bullet$  the set of output places of a transition  $t$  and  $\bullet\tilde{T} := \{\bullet t \mid t \in \tilde{T}\}$ ,  $\tilde{T}\bullet := \{t\bullet \mid t \in \tilde{T}\}$

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 $T_H := T_2$ 
while  $\bullet T_H \neq T_H \bullet$  do
begin
   $S_H := (\bullet T_H \setminus T_H \bullet) \cup (T_H \bullet \setminus \bullet T_H)$ 
   $\forall t \in T_H$  do
    if  $(\bullet t \cap S_H) \cup (t \bullet \cap S_H) \neq \emptyset$ 
    then  $T_H := T_H \setminus \{t\}$ 
end
if  $T_H = \emptyset$ 
then Condition NoTT is satisfied
else Condition NoTT is not satisfied

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Figure 5: Algorithm for testing condition NoTT

## 5 Conclusions

The avoidance of timeless traps is a basic requirement for the extraction of a Markov process description from a GSPN. Condition NoTT is a sufficient condition for doing so. Although it is not necessary, it hints at critical subnets. Furthermore the elimination of vanishing states during state space generation is straight forward for GSPNs satisfying condition NoTT.

## References

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