Analysis of Petri Nets with a Dynamic Priority Method

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Abstract. In this paper we propose an efficient reachability set based Petri net analysis by introducing dynamic priorities which decreases the number of reachable markings in most cases. It is proved that for specific dynamic priority relations certain properties (especially liveness and the existence of home states) do hold if and only if these properties do also hold for the Petri net without priorities. We present an algorithm based on these priority relations additionally exploiting T-invariants thus being able to benefit from the structure of the net.

1 Introduction

Analysis of Petri nets by exploring the set of reachable markings is a straightforward approach, since most properties are defined on the basis of the reachability graph[12]. Although this kind of analysis of Petri nets suffers from the well-known state space explosion problem, it is offered by most analysis tools[7], because it is easy to implement and in case of error detection appropriate traces (firing sequences) are also easy to derive. Furthermore they can be applied at least in principal to general (bounded) Petri nets, whereas efficient structural methods have their limits, e.g. the analysis of special net classes [6].

Methods trying to cope with the state space explosion problem generate a compressed representation of the whole reachability graph (e.g. exploiting symmetries or structured representations[10]) or reduce the size of the reachability graph without affecting the relevant properties under consideration[8, 14, 17, 19]. Reduction of the size is a suitable approach when we are only interested in "global functional" properties, which is often the case.

In this article we use dynamic priorities to reduce the number of markings in the reachability graph. The idea is similar to that of the stubborn set method[20]: we want to reduce the number of enabled transitions at a given marking thus hoping to reduce the size of the reachability graph significantly. Several authors mentioned that the stubborn set method can also be considered a dynamic priority method (see e.g. [21]), but to the best of our knowledge this method has not been investigated further in this context.

Since we are interested in specific properties of the original Petri net, we can not use arbitrary priority relations. In Sec. 3 we define an appropriate class of priority relations which leave several properties of the net unaffected. The results of this section are an extension of those given in [1] for static priorities, where only one direction was considered starting with the assumption that the Petri net satisfies the properties of interest. We also show that the set of enabled transitions at each marking according to the proposed dynamic priority relations can be viewed as stubborn sets, but that ignoring does not occur, so that also further properties can be checked, where we have concentrated on those properties which are also important in the context of stochastic Petri nets[2]. Furthermore the proposed dynamic priority relations also allow the investigation of specific markings, e.g. deciding whether the initial marking is a home state.

In Sect. 4 we discuss an algorithm based on dynamic priorities also exploiting information on the T-invariants of the net. By means of examples we demonstrate the algorithms efficiency, showing that the reduced reachability graph is of moderate size.

2 Basic definitions

This section recalls basic notions of Place/Transition nets (P/T-nets) with weighted arcs (cf. [12]) and some definitions for relations.

Definition 1. A **Place/Transition net** (P/T-net) is a triple $N = (P, T, W)$ with $P \cap T = \emptyset$ and $W : ((P \times T) \cup (T \times P)) \mapsto \mathbb{N}_0$. P is a finite set of places and T a finite set of transitions. W specifies the interconnection of places and transitions.

For $x \in P \cup T$ the *preset* is given by $\bullet x = \{y \in P \cup T \mid W(y,x) > 0\}$ and the *postset* by $x\bullet = \{y \in P \cup T \mid W(x,y) > 0\}$ and the usual extension to sets $X \subseteq P \cup T$ is defined as $\bullet X = \bigcup_{x \in X} \bullet x$, $X\bullet = \bigcup_{x \in X} x\bullet$. A *marking* of a P/T-net is a function $M : P \mapsto \mathbb{N}_0$, where $M(p)$ denotes the number of tokens in p. A P/T-net N with an initial marking M_0 is called a *P/T-system*, denoted by (N, M_0). A set $\tilde{P} \subseteq P$ is *marked* in a marking M, iff $\exists p \in \tilde{P} : M(p) > 0$; otherwise $\tilde{P}$ is *unmarked* or *empty* in M. A transition $t \in T$ is *enabled* at M, denoted by $M[t >$, iff $M(p) \geq W(p,t), \forall p \in P$. $EN(M) := \{t \in T | M[t >\}$ is the set of enabled transitions at M. A transition $t \in T$ being enabled at marking M *may fire* yielding a new marking given by $M'(p) = M(p) - W(p,t) + W(t,p), \forall p \in P$, denoted by $M[t > M'$. A *P-invariant* is a vector $v = (v_1, \ldots, v_{|P|})$ satisfying $\sum_{i=1}^{|P|} v_i * (W(t,p_i) - W(p_i,t)) = 0, \forall t \in T$. A *T-invariant* is a vector $v = (v_1, \ldots, v_{|T|})$ satisfying $\sum_{i=1}^{|T|} (W(t_i,p) - W(p,t_i)) * v_i = 0, \forall p \in P$.

The *reachability set* $[M_0 >$ is the smallest set satisfying $M_0 \in [M_0 >$ and if $M' \in [M_0 >, M'[t > M''$ for some $t \in T$ then $M'' \in [M_0 >$. As usual, the *reachability graph* comprises all markings of the reachability set as nodes and the corresponding enabled transitions as edges. $\tilde{R} \subseteq [M_0 >$ is a *final strongly connected component* of $[M_0 >$ iff $\forall M \in \tilde{R}, M' \in [M_0 >: M' \in [M > \Longrightarrow (M \in [M' >$ and $M' \in \tilde{R})$. A *firing sequence* of N is a finite sequence of transitions $\sigma = t_1 \ldots t_n, n \geq 0$ such that there are markings $M_1, \ldots, M_{n+1}$

satisfying $M_i[t_i > M_{i+1}, \forall i = 1, \ldots, n$. A shorthand notation for this case is $M_1[\sigma >$ and $M_1[\sigma > M_{n+1}$ resp. σ has *concession* at M iff $M[\sigma >$ holds. ε denotes the empty firing sequence and $M[\varepsilon > M$ holds for all markings M. If a transition t is part of a firing sequence σ we denote this by $t \in \sigma$. $\#(\sigma, t)$ denotes the number of occurrences of t in σ and $|\sigma|$ denotes the length of σ. (N, M_0) is *safe*, iff $\forall p \in P : \forall M \in [M_0 >: M(p) \leq 1$. (N, M_0) is *bounded*, iff $\forall p \in P : \exists k \in \mathbb{N}_0 : \forall M \in [M_0 >: M(p) \leq k$. (N, M_0) is *live*, iff $\forall t \in T, M \in [M_0 >: \exists M' \in [M >: M'[t >$. (N, M_0) is *dead*, iff $\exists M \in [M_0 >: EN(M) = \emptyset$. A marking $M \in [M_0 >$ is a *home state*, iff $\forall M' \in [M_0 >: M \in [M' >$. (N, M_0) is *reversible*, iff M_0 is a home state. N is an *extended free choice net (EFC-net)*, iff $\forall p \in P, t \in T : W(p,t) \leq 1, W(t,p) \leq 1$ and $\forall t, t' \in T : \bullet t \cap \bullet t' = \emptyset$ or $\bullet t = \bullet t'$. N is an *equal conflict net (EC-net)*, iff $\forall t, t' \in T : \bullet t \cap \bullet t' = \emptyset$ or $W(p,t) = W(p,t'), \forall p \in P$. $SCS := \{(t,t') \in T \times T | \bullet t \cap \bullet t' \neq \emptyset\}$ is the relation of structural conflict sets. SCS^* is the reflexive and transitive closure of the relation SCS and with $[t]_{SCS^*}$ we denote the equivalence class t belongs to. $T_{EC} := \{t \in T | \forall t' \in (\bullet t)\bullet : W(p,t) = W(p,t'), \forall p \in P\}$ is the set of transitions which locally exhibit an equal conflict net structure and $T_{NEC} := T \setminus T_{EC}$.
In the following we will call P/T-nets also Petri nets.

Definition 2 relations (cf. [9]). Let T denote some set. $\rho \subseteq T \times T$ is a *relation*. ρ is *reflexive* iff $(t,t) \in \rho, \forall t \in T$. ρ is *irreflexive* iff $(t,t) \notin \rho, \forall t \in T$. ρ is *symmetric* iff $(t,t') \in \rho \Rightarrow (t',t) \in \rho$. ρ is *asymmetric* iff $(t,t') \in \rho \Rightarrow (t',t) \notin \rho$. ρ is *transitive* iff $(t,t'), (t',t'') \in \rho \Rightarrow (t,t'') \in \rho$. ρ is an *equivalence relation* iff ρ is reflexive, symmetric and transitive. If ρ is an equivalence relation, then for $t \in T$, $[t]_\rho$ denotes the *equivalence class* t belongs to. $\rho^{-1} := \{(t,t') \in T \times T \mid (t',t) \in \rho\}$. $\overline{\rho} := \{(t,t') \in T \times T \mid (t,t') \notin \rho \cup \rho^{-1}\}$.

In the following we do only consider irreflexive, asymmetric and transitive relations ρ for which $\overline{\rho}$ is an equivalence relation and which take all elements of T into account, i.e. $\{t \in T | \exists t' \in T : (t,t') \in \rho$ or $(t',t) \in \rho\} = T$. Trivially reflexive, symmetric or non-transitive relations will affect analysis of liveness aspects negatively (cf. [1]). Assuming that $\overline{\rho}$ is an equivalence relation will simplify the further discussion, since the equivalence classes of $\overline{\rho}$ determine transitions of the same priority level/class.

3 Petri nets with dynamic priorities

A common method of defining priorities for P/T-nets is to impose an additional relation $\rho \subseteq T \times T$ on the set of transitions[3]. Dynamic priorities (cf. [11]) are usually defined in dependence on the current marking.

Definition 3. A *priority P/T-net* is a pair (N, ρ) where N is a P/T-net and $\rho \in [\mathbb{N}_0^{|P|} \mapsto \subseteq T \times T]$ is a function defined on $\mathbb{N}_0^{|P|}$ defining a priority relation for each possible marking of the net.

The dynamics of such a priority P/T-net is characterised by an enabling rule taking these priorities into account.

Definition 4. A transition $t \in T$ is ρ-enabled at a marking M, denoted by $M[t>_\rho$, iff $M[t>$ and $\forall t' \in T : M[t'> \Longrightarrow (t,t') \notin \rho(M)$.

Note that dynamic priorities are a generalisation of static priorities where the priority relation is fixed for all markings.

All other definitions given in Sect. 2 can be easily reformulated for priority P/T-nets in the context of ρ-enabling. Similar to [1] we will prefix or subscript all notions for priority P/T-nets with "ρ", e.g. we will denote the ρ-enabling of a firing sequence $f \in T^*$ at a marking $M \in \mathbb{N}_0^{|P|}$ by $M[f>_\rho$. In the following (N,ρ,M_0) is a P/T-system with priorities and (N,M_0) is the corresponding underlying P/T-system without priorities.

Using priorities for the purpose of analysing the underlying Petri net is surely limited to specific functions ρ. Consider the trivial example of Fig. 1. If we define, e.g., ρ such that $(t',t) \in \rho(M)$ for all markings M then the priority P/T-net would have a smaller reachability set than the original net. Obviously, when considering liveness this priority relation would be of no use for us, since the original net is live, whereas the priority P/T-net is not. To cope with such situations an obvious idea is to define that all transitions of a structural conflict set should belong to the same priority class. In [1] a net with static priorities (see Fig. 2) is presented. Here the reachability graph of the priority P/T-net (cf. Fig. 5) would not allow checking liveness of (N,M_0), although all transitions of a structural conflict set have the same priority. This problem does not occur in the example net of Fig.2, when we demand for all transitions of T_{NEC} to belong to the lowest priority class. If we are interested in further properties especially those concerning specific markings additional restrictions need to be imposed on the priority relation. In the net of Fig. 4 we have $T_{NEC} = \{t_1,t_4\}$ and Fig. 5 shows the reachability graphs of (N,M_0) and (N,ρ,M_0) where T_{NEC} belongs to the lowest priority class and t_3 always has priority on all other transitions. If one is now interested in whether M_0 is a home state, the information given by $[M_0>_\rho$ is not sufficient. A priority relation where amongst T_{NEC} additionally all transitions enabled at, in this case, M_0 belong to the lowest priority class (i.e. also t_3 should have a low priority; cf. Fig 5) would lead to a situation where M_0 is a home state of (N,ρ,M_0) as well.

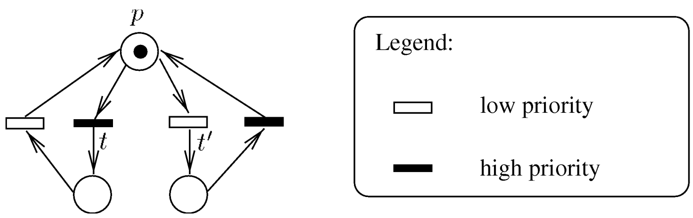

Fig. 1. Trivial example for liveness (static priorities)

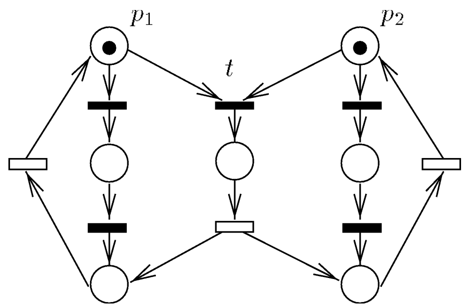

Fig. 2. Live P/T-net; Non-live with static priorities

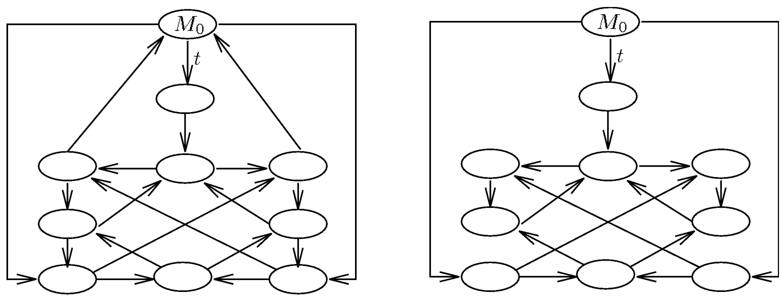

Fig. 3. Reachability graphs: PN (left); PN with priorities (right)

We will exploit the above developed ideas now for the dynamic priority case, heading for those priority relations which ensure that specific properties of the original Petri net do also hold for the net with priorities and vice versa.
For an arbitrary, but fixed (possibly empty) $S \subseteq T$ define

$$T^S_{low} := T_{NEC} \cup \bigcup_{t \in S} [t]_{SCS^*} \tag{1}$$

and for $M \in \mathbb{N}_0^{|P|}$ define

$$\Omega_M := \{t \in T | \forall t' \in T : (t', t) \notin \rho(M)\} \tag{2}$$

i.e. Ω_M is the set of transitions with the lowest priority at marking M. T^S_{low} comprises all transitions of T_{NEC} and additionally SCS^*-sets of transitions from T_{EC}. Note that $T^{\emptyset}_{low} = T_{NEC}$.

T^S_{low} denotes the set of transitions which have to belong to the lowest priority class. As we have realised for the net of Fig. 2 all transitions of T_{NEC} should

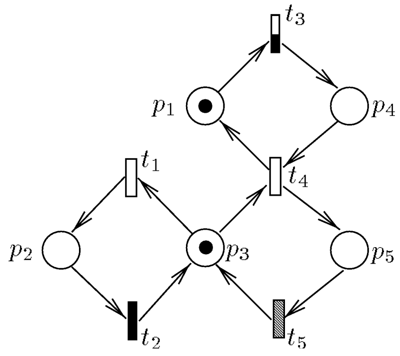

Fig. 4. M_0 home state? (t_5 medium priority)

be in T^S_{low} (i.e. $T_{NEC} \subseteq T^S_{low}$) and considering the net of Fig. 4 we have seen that additional transitions should be included as well. From our observation (cf. Fig. 1) that transitions in structural conflict should belong to the same priority class, all other transitions of $[t]_{SCS^*}$ should thus also be included.

Furthermore, now with dynamic priorities in mind, it is obvious that the priority levels for transitions of $T_{EC} \setminus T^S_{low}$ need not be the same at all markings.

In summary the following restriction seems to be a good candidate to ensure that the reachability graph of the priority P/T-net can be used to check properties of the underlying P/T-net.

Definition 5 Condition DEC.
A priority P/T-net satisfies condition *dynamically EQUAL-Conflict (DEC)* iff

$$\forall M \in \mathbb{N}_0^{|P|} : \forall t \in T : M[t>_\rho \implies \text{a) } t \in T \setminus T^S_{low} \implies \forall t' \in [t]_{SCS^*} : M[t'>_\rho \quad \underline{\text{and}}$$
$$\text{b) } t \in T^S_{low} \implies \Omega_M = T^S_{low}$$

In the following we prove that condition DEC ensures that liveness and the existence of home states can be verified on the basis of the reachability graph of the priority P/T-net.

The concept of ρ-enabling restricts the firing sequences of a P/T-net, in particular all firing sequences of the priority P/T-net are also firing sequences of the underlying P/T-net but not vice versa. Thus the reachability set of (N, ρ, M_0) is a subset of the reachability set of (N, M_0).

Lemma 6. *[1] Let $M : P \mapsto \mathbb{N}_0, t \in T, f \in T^*$.*

1. $M[f>_\rho \implies M[f>$
2. $[M>_\rho \subseteq [M>$
3. $M[t> \implies \exists t' \in T : M[t'>_\rho$

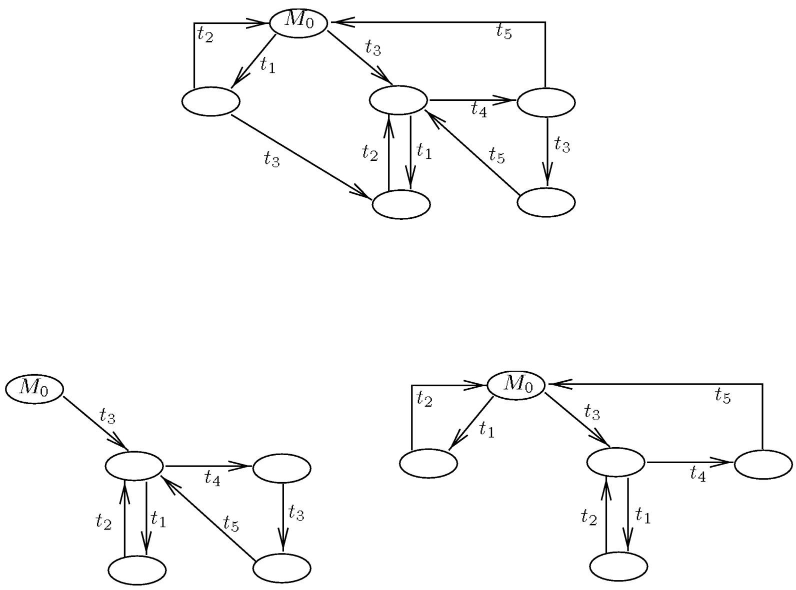

Fig. 5. Reachability graphs: original Petri net (top), t_3 high priority (bottom left), t_3 low priority (bottom right)

Lemma 6(3) states that for each marking with an enabled transition we have at least one ρ-enabled transition, which follows from the restrictions imposed on $\rho(M)$ at the end of Sect. 2.

The next lemma establishes the basis of the priority relations considered in this paper. It merely states that a firing sequence starting at some marking M ($\in [M_0 >_\rho$) and ending at a marking M' where only transitions of T^S_{low} are enabled has a permuted counterpart having ρ-concession at M. This implies that M' is in $[M_0 >_\rho$ as well.

Lemma 7 (cf. [1]). *Let (N, ρ) satisfy condition DEC. Let M, M' be some markings and $g \in T^*$ be a firing sequence such that $M[g > M'$ and*

$$\forall t \in T : M'[t > \Longrightarrow t \in T^S_{low} . \qquad (3)$$

Then $\exists \tilde{g} \in Perm(g) : M[\tilde{g} >_\rho$.

where $Perm(g)$ denotes the set of all permutations of g. The proof of lemma 7 is given in the appendix.

Note that all deadlock markings of (N, M_0), i.e. markings satisfying $EN(M) = \emptyset$, trivially satisfy (3). Thus this lemma directly shows that all deadlock states of $[M_0 >$ are also in $[M_0 >_\rho$.

Markings in $[M_0 >$ satisfying (3) are thus also in $[M_0 >_\rho$. Obviously those markings will be helpful, when considering properties of the Place/Transition-net with and without priorities. Thus it is desirable that such markings do really exist. The main problem is the case, when $T^{\emptyset}_{low} = \emptyset$, because then only deadlock markings satisfy (3). In the following we will assume $T^S_{low} \neq \emptyset$. The discussion of the empty set-case, implying an EC-net, will be postponed to the next subsection.

Lemma 7 is only applicable if we can ensure that the P/T-net will eventually reach a marking satisfying (3). It might happen that transitions of T^S_{low} are ignored, i.e. only higher priority transitions are fired. For the priority P/T-net this implies that it is caught in a trap. Fig. 6 shows an example of a ρ-trap which is constituted by the firing of t and t'. Lemma 9 shows that this situation does not occur if condition DEC holds.

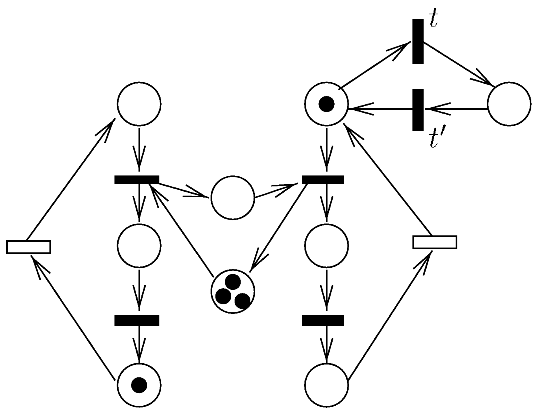

Fig. 6. Example of a ρ-trap

Definition 8 **ρ-trap.** (N, ρ, M_0) has a *ρ-trap* iff $\exists M \in [M_0 >$ with

1. $\forall k \in \mathbb{N} : \exists g \in T^* : |g| \geq k$ and $M[g >_\rho$ and
2. $\forall f \in T^* : (\ M[f >_\rho \Longrightarrow f \in (T \setminus T^S_{low})^* \ )$.

Lemma 9. *Let N be strongly connected and (N, M_0) be bounded. If (N, ρ) satisfies condition DEC, then (N, ρ, M_0) has no ρ-trap.*

Proof. Assume (N, ρ, M_0) has a ρ-trap. Since (N, M_0) is bounded, $[M' >_\rho$ is finite $\forall M' \in [M_0 >$ (cf. lemma 6), thus $[M >_\rho$ contains a final strongly connected component $C_M \subseteq [M >_\rho$. Define $\tilde{T} := \{t \in T | \exists M' \in C_M : M'[t >_\rho\}$ and $\tilde{P} := \bullet\tilde{T}$. $\tilde{T}$ can be regarded as the set of ρ-live transitions with respect to C_M. Since N is strongly connected and because of our assumption of the existence of a ρ-trap implying $\tilde{T} \neq \emptyset$ and $(T \setminus \tilde{T}) \neq \emptyset$ (note that in this case also $T_{NEC} \subseteq T^S_{low} \subseteq T \setminus \tilde{T}$ and $T_{EC} \subseteq \tilde{T}$), one of the following two cases holds:

a: $\exists p_r \in \tilde{P} : p_r\bullet \supseteq \{t_i, t_j\}$ and $t_i \in \tilde{T}, t_j \in T \setminus \tilde{T}$.
Since condition DEC holds, part a) implies that if $M'[t_i >_\rho$ for some $M' \in C_M$, then $M'[t_j >_\rho$, contradicting $t_j \notin \tilde{T}$.

b: $\exists t_r \in \tilde{T} : t_r\bullet \supseteq \{p_k, p_l\}$ and $p_k \in \tilde{P}, p_l \in P \setminus \tilde{P}$.
Since $t_r \in \tilde{T}$, t_r might fire infinitely often with regard to C_M. Since $p_l \notin \tilde{P} = \bullet\tilde{T}$, place p_l is not bounded, contradicting boundedness of (N, M_0).

Thus assuming the existence of a ρ-trap was wrong which completes the proof.
□

Lemma 9 actually shows that we can always reach a marking $M \in [M_0 >$ satisfying (3). E.g. if we have a dead net the deadlock marking trivially satisfies (3). If we do not reach a dead marking, then there is always a firing transition and thus the second part of definition 8 does not hold, i.e. we always reach a marking M' with $\exists t \in T^S_{low} : M'[t >_\rho$. Since $[M_0 >_\rho \subseteq [M_0 >$ and condition DEC holds, M' satisfies (3).

Now we can use lemmas 7 and 9 to consider liveness and the existence of home states for priority P/T-nets satisfying condition DEC.

Theorem 10. *Let N be strongly connected, (N, ρ) satisfying condition DEC and (N, M_0) be bounded. Then the following holds:*

1. *(N, ρ, M_0) is ρ-bounded.*
2. *(N, M_0) dead $\Longleftrightarrow$ (N, ρ, M_0) ρ-dead*
3. *(N, M_0) live $\Longleftrightarrow$ (N, ρ, M_0) ρ-live*
4. *(N, M_0) has home states $\Longleftrightarrow$ (N, ρ, M_0) has ρ-home states*
5. *Let M be a marking of (N, ρ) satisfying (3) then*
   M is a home state of (N, M_0) $\Longleftrightarrow$ M is a ρ-home state of (N, ρ, M_0)

Proof. **(1)** Obvious, since $[M_0 >_\rho \subseteq [M_0 >$.

(2) **"$\Longrightarrow$":** (N, M_0) dead implies the existence of a marking M and a firing sequence $g \in T^* : M_0[g > M$ and $EN(M) = \emptyset$. Since M satisfies (3) we have $M \in [M_0 >_\rho$ as well (cf. lemma 7) and thus (N, ρ, M_0) is ρ-dead, since $EN_\rho(M) \subseteq EN(M)$ by definition.

"$\Longleftarrow$:" (N, ρ, M_0) dead implies the existence of $M \in [M_0 >_\rho \subseteq [M_0 >$ with $EN_\rho(M) = \emptyset$. Because of the conditions we imposed at the end of Sect. 2 on $\rho(M)$, we have $EN(M) = \emptyset$ (cf. also lemma 6(3)), i.e. (N, M_0) is dead.

(3) **"$\Longrightarrow$":** Assume (N, ρ, M_0) is not live, i.e. $\exists M \in [M_0 >_\rho$: $\exists t \in T : \forall M' \in [M >_\rho$: $\neg M'[t >_\rho$. (N, M_0) being live and bounded implies (N, M) is live and bounded, since $M \in [M_0 >_\rho \subseteq [M_0 >$. Together with lemma 9 this implies the existence of $g, h \in T^*$ and markings $\tilde{M}, \tilde{M}' \in [M >$ with $M[g > \tilde{M}[th > \tilde{M}'$ such that $\tilde{M}'$ satisfies (3). Employing lemma 7 there exists $\tilde{g} \in Perm(gth)$ with $M[\tilde{g} >_\rho$ contradicting our assumption.

"$\Longleftarrow$": Assume (N, M_0) is not live. I.e. $\exists M \in [M_0>: \exists t \in T : \forall M' \in [M>: \neg M'[t>$. Now (N, M) is bounded as well and so the preconditions of lemma 9 are satisfied, implying the existence of $\tilde{M} \in [M > \subseteq [M_0 >$ satisfying (3) yielding $\tilde{M} \in [M_0 >_\rho$. From our assumption we have $\forall M' \in [\tilde{M} >: \neg M'[t >$ implying $\forall M' \in [\tilde{M} >_\rho \subseteq [\tilde{M} >: \neg M'[t >_\rho$ contradicting $\rho-$liveness of (N, ρ, M_0).

(4) The existence of home states is equivalent to the directedness property [4] which is

$$\forall M, M' \in [M_0 >: [M > \cap [M' > \neq \emptyset$$

or in terms of priority P/T-nets:

$$\forall M, M' \in [M_0 >_\rho: [M >_\rho \cap [M' >_\rho \neq \emptyset.$$

"$\Longrightarrow$": So, choosing arbitrary markings $M, M' \in [M_0 >_\rho$, there exists a marking $M'' \in [M > \cap [M' >$, since (N, M_0) has home states. Let $f, g \in T^*$ denote the corresponding firing sequences, i.e. $M[f > M''$ and $M'[g > M''$. Again lemma 9 ensures the existence of $\tilde{M}'' \in [M'' >_\rho$ with $\tilde{M}''$ satisfying (3). Lemma 7 finally implies $\tilde{M}'' \in [M >_\rho$ and $\tilde{M}'' \in [M' >_\rho$, i.e. $\tilde{M}'' \in [M >_\rho \cap [M' >_\rho$.

"$\Longleftarrow$": Assume (N, M_0) has no home states. Thus $\exists M, M' \in [M_0 >: [M > \cap [M' >= \emptyset$. Using lemma 9 we can find markings $\tilde{M} \in [M >$ and $\tilde{M}' \in [M' >$ both satisfying (3). Thus lemma 7 implies $\tilde{M} \in [M_0 >_\rho$ and $\tilde{M}' \in [M_0 >_\rho$. Since (N, ρ, M_0) has $\rho-$home states we have $[\tilde{M} >_\rho \cap [\tilde{M}' >_\rho \neq \emptyset$ contradicting our assumption.

(5) "$\Longrightarrow$": M being a home state of (N, M_0) means (cf. Def. 1) $\forall M' \in [M_0 >: M \in [M' >$. Since $[M_0 >_\rho \subseteq [M_0 >$ we have $\forall M' \in [M_0 >_\rho: M \in [M' >$ and because of lemma 7 we further have $\forall M' \in [M_0 >_\rho: M \in [M' >_\rho$.

"$\Longleftarrow$": M being a $\rho-$home state of (N, ρ, M_0) means $\forall M' \in [M_0 >_\rho: M \in [M' >_\rho$. Now assume that M is not a home state of (N, M_0), i.e. $\exists M' \in [M_0 >: M \notin [M' >$. Since (N, M_0) is bounded and $M' \in [M_0 >$ there is a final strongly connected component $C_{M'} \in [M' >$. Lemma 9 yields the existence of $\tilde{M} \in C_{M'}$ satisfying (3) and thus by lemma 7 again $\tilde{M} \in [M_0 >_\rho$. Since M is a $\rho-$home state there exists $f \in T^*$ with $\tilde{M}[f >_\rho M$. Thus $M \in [\tilde{M} >$ giving $M \in [M' >$ contradicting our assumption. □

Part 5 of theorem 10 is, e.g., of interest if we want to check for reversibility. In that case we only have to define ρ such that M_0 satisfies (3), i.e. $EN(M_0) \subseteq T^S_{low}$. Although one should notice that in the worst case we might get $[M_0 >= [M_0 >_\rho$, giving no performance improvement for the reachability analysis.

The set $EN_\rho(M)$ given by dynamic priorities satisfying condition DEC can be viewed as a special case of the transition sets defined by the stubborn set method [18, 19]. This follows directly from theorem 2.7 in [18], which simplifies the stubborn set definition for nets without self-loops:
Assuming nets without self-loops, i.e. $\bullet t \cap t \bullet = \emptyset, \forall t \in T$ a subset $T_s \subset T$ is semistubborn at M, iff $\forall t \in T_s$:

$$\exists s \in \bullet t : M(s) < W(s,t) \text{ and } \bullet s \subseteq T_s$$
$$\text{or } (\forall s \in \bullet t : \bullet s \subseteq T_s \text{ or } (s\bullet \subseteq T_s \text{ and } M(s) \geq W(s,t))).$$

T_s is stubborn at M, iff T_s is semistubborn at M and
$\exists t \in T_s : \forall s \in \bullet t : M(s) \geq W(s,t)$ and $s\bullet \in T_s$.

If $EN_\rho(M) \cap T^S_{low} = \emptyset$ then the set $EN_\rho(M)$ is stubborn at M, since $\forall t \in EN_\rho(M)$ we have $\forall s \in \bullet t : s\bullet \subseteq EN_\rho(M)$ and trivially $M(s) \geq W(s,t)$. And when $EN_\rho(M) \cap T^S_{low} \neq \emptyset$ holds, then $\Omega_M = T^S_{low}$ implies $EN_\rho(M) = EN(M)$ and we can select all transitions of the net as the stubborn set.

In contrast to the basic stubborn set method, dynamic priorities satisfying condition DEC do not suffer from the ignoring problem as shown in lemma 9 and furthermore specific markings can be investigated (see Th. 10 part 5).

3.1 EC-nets with dynamic priorities

Let us now consider the case $T^{\emptyset}_{low} = \emptyset$, which implies that the net is an EC-net (cf. (1)). Condition DEC now simplifies to

Definition 11 Condition DEC for EC-nets.
A priority EC-net satisfies condition *dynamically EQUAL-Conflict (DEC)* iff
$\forall M \in \mathbb{N}_0^{|P|} : \forall t \in T : M[t>_\rho \Longrightarrow \forall t' \in [t]_{SCS^*} : M[t'>_\rho$

Theorem 12. *Let N be a strongly connected EC-net, (N,ρ) satisfying condition DEC and (N,M_0) be bounded. Then the following holds:*

1. *(N,M_0) not dead* $\Longleftrightarrow^{(a)}$ *(N,M_0) live* $\Longleftrightarrow^{(b)}$ *(N,ρ,M_0) ρ-live* $\Longleftrightarrow^{(c)}$ *(N,ρ,M_0) not ρ-dead*
2. *(N,ρ,M_0) has ρ-home states* $\Longrightarrow$ *(N,M_0) has home states*

Proof. **(1) (a):** see, e.g. [16].

(c): "$\Longrightarrow$": obvious.

"$\Longleftarrow$": (similar to (a)) Assume (N,ρ,M_0) is not $\rho-$live, i.e. $\exists M \in [M_0>_\rho$: $\exists t \in T : \forall M' \in [M>_\rho$: $\neg M'[t>_\rho$. Because of condition DEC and the EC-net structure, all transitions of $[t]_{SCS^*}$ are dead as well, i.e. $\forall t' \in [t]_{SCS^*} : \forall M' \in [M>_\rho$: $\neg M'[t>_\rho$. Since (N,ρ,M_0) is $\rho-$bounded (cf. theorem 10(1)) all transitions of $\bullet(\bullet[t]_{SCS^*})$ are also not $\rho-$live, i.e. $\exists \tilde{M} \in [M>_\rho$: $\forall t' \in \bullet(\bullet[t]_{SCS^*}) : \neg\tilde{M}[t'>_\rho$. Repeating this argument and thanks to the strong connectedness of N finally all transitions will be dead contradicting the $\rho-$deadlock-freeness of (N,ρ,M_0).

(b): Finally we show: (N,M_0) not dead $\Longleftrightarrow^{(d)}$ (N,ρ,M_0) not ρ-dead giving us (b) implicitly. Now (d) is directly implicated by lemma 7 and Th. 10(2), since all deadlock markings of $[M_0>$ are in $[M_0>_\rho$ and vice versa. That completes the proof of part (1).

(2) Here we can distinguish between two cases: (N, M_0) being live and being non-live. If the net is live then together with boundedness the existence of home states is guaranteed ([16]), so we have nothing to show. Therefore let us assume that (N, M_0) is non-live. This implies that (N, M_0) is dead (see part (1)). Since all deadlock markings of $[M_0>$ are part of $[M_0>_\rho$ as well, the theorem follows. □

4 Petri net analysis employing dynamic priorities

Using the results of the former sections might give us an efficient analysis procedure based on the usually smaller reachability set of the net with dynamic priorities. The main problem is to define a suitable priority function ρ. A first, trivial idea is that we can define such a priority relation on the fly during reachability set generation. From the definition of DEC we know that at a marking M where a transition $t \in T_{EC} \setminus T^S_{low}$ is enabled we can always select the complete set $[t]_{SCS^*}$ for firing, which is enabled at M as well, i.e. we can assign the highest priority to $[t]_{SCS^*} \subseteq EN(M)$ and act analogous. Last not least, when all former considered cases, which correspond to condition DEC part a) can not be applied, we might select all transitions of $EN(M)$ for ρ-firing thus satisfying part b) of condition DEC. E.g. in the simple Petri net of Fig. 7 we could fire all transitions in sequence starting with t_1 and ending with t_N thus having implicitly defined a dynamic priority relation satisfying condition DEC. For this special net we could therefore manage to generate a reachability set $[M_0>_\rho$ of size N independent of the initial marking.

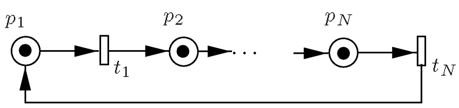

Fig. 7. Simple chain

k	$\vert[M_0>\vert$	$\vert[M_0>_\rho\vert$	stubborn
1	92378	10	54
2	10015005	10	99
3	2.119e+08	10	144
4	2.054e+09	10	189
5	1.257e+10	10	234
10	4.263e+12	10	459
15	1.421e+14	10	684
20	1.761e+15	10	909

Fig. 8. Reachability set sizes for the chain with $N = 10$ and $M_0 = (k, \ldots, k)$.

Since the transitions determined by dynamic priorities satisfying condition DEC can be viewed as special cases of stubborn sets we also present the sizes of the reachability sets for $N = 10$ obtained with the stubborn set method using the incremental algorithm of the tool PROD [13] (see Fig. 8). The parameter k determines the number of tokens for all places p_i at the initial marking.

The intention of these comparisons is to give an impression on the efficiency of the algorithm we are going to develop.

The key issue in reducing the size of the reachability set is to select the appropriate SCS^*-set of transitions amongst all such enabled sets. In general as indicated by this trivial example the size of $[M_0 >_\rho$ might be small if we only fire each transition (respectively SCS^*-set) as often as determined by positive T-invariants. Although the worst case complexity of invariant calculation is exponential, from a practical point of view the computing time is often moderate.

In the following let us assume that the net is covered by positive P- and T-invariants, thus ensuring boundedness and satisfying a necessary condition for liveness, and that a cover of positive T-invariants is given explicitly. The idea of firing T-invariants leads to a straightforward algorithm for marked graphs. For general net structures such an algorithm is not so simple to find, since condition DEC forces us to fire SCS^*-sets of transitions. If we assume a depth first search approach during generation of the reachability graph one approach is to fire all SCS^*-sets as often as specified by the corresponding T-invariants. Since T-invariants take only single transitions into account, we have to define some "invariant weight" $INV(V,Q)$ for a subset $Q \subseteq T$ given an (appropriate) set V of positive T-invariants covering the net. Given $INV(V,Q)$ we can exploit this information as follows: Build up a list of SCS^*-sets for transitions in $T \setminus T^S_{low} \subseteq T_{EC}$, where each SCS^*-set Q has been inserted as often as given by $INV(V,Q)$ and use this list as proposed in the algorithm of Fig. 9. Since each SCS^*-set Q occurs as often as specified by $INV(V,Q)$ in the corresponding list, the algorithm in a way mimics the firing of T-invariants and should perform well when greater parts of the net locally have an EC-net structure.

```
Calculate EN_ρ(M) in each generation step of the reachability analysis as follows:
Let Q_1,...,Q_r be the equivalence classes of SCS* for transitions of T \ T^S_low.
Define Q̂ := ⋃_{i=1}^r {Q_i | Q_i ⊆ EN(M)}
if Q̂ = ∅ return EN(M)
else
begin
    find first element e in list_EC with e ∈ Q̂
    remove e from list_EC and insert it at the end of list_EC
    return e
end
```

Fig. 9. An algorithm for determining $EN_\rho(M)$.

There are some problems still to solve. Firstly how to initialise the list and secondly how to define, amongst finding V, $INV(V,Q)$ appropriately. Additionally the most general problem with this approach is that during reachability graph generation the exploration of other branches has to be postponed so that we have to store the information which part of the T-invariant has been already fired. Obviously this would increase the overall storage requirement for this approach tremendously, since in the worst case this information has to be stored at all markings. Therefore we propose the heuristics, that the list is kept global, hoping that once we have fired the transitions of a T-invariant, the (ordering of the) list is still helpful for other, currently non-explored markings.

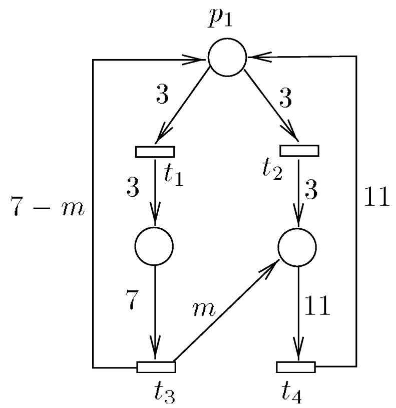

Fig. 10. EC-net with $M_0 = (31, 0, 0)$

Considering an appropriate definition of $INV(V, Q)$ let us consider the net of Fig. 10 first with $m = 0$ to get some insight in what might happen. A set of positive T-invariants is e.g. $V = \{(7, 0, 3, 0), (0, 11, 0, 3)\}$. According to condition DEC, e.g., t_1 and t_2 should be both ρ-enabled at M_0 (provided they are enabled at M_0). So we have to define $INV(V, \{t_1, t_2\})$. The problem is now that the invariant weights of the individual transitions differ, e.g. we can choose $INV(V, \{t_1, t_2\}) = 7$ giving $|[M_0 >_\rho| = 338$ or $INV(V, \{t_1, t_2\}) = 11$ giving $|[M_0 >_\rho| = 471$. (We assume that initially $\{t_4\}$ and $\{t_3\}$-elements occur before $\{t_1, t_2\}$-elements in the list.)

On the other hand one can also select $V = \{(77, 77, 33, 21)\}$ (a linear combination of the former ones) yielding $|[M_0 >_\rho| = 131$ thus indicating that it is more appropriate to select V such that all transitions of a SCS^*-set have the same weight. Of course this is not always possible. If we define $m = 4$ in the net of Fig. 10 a set of minimal T-invariants is $V = \{(0, 11, 0, 3), (77, 0, 33, 12)\}$. Adjusting the weight of t_2, i.e. defining $V' = \{(0, 77, 0, 21), (77, 0, 33, 12)\}$ does not determine a unique weight for t_4. Choosing $INV(V, \{t_4\}) = 12$ results in a reachability set size $|[M_0 >_\rho| = 315$, whereas $INV(V, \{t_4\}) = 21$ gives us $|[M_0 >_\rho| = 277$. From that point of view it seems to be more appropriate to choose the maximum, since then the chances are better that a T-invariant can be completed before other transition firings interfere, which themselves have to be recognised for completion of further T-invariants later on.

Definition 13. If V is a set of positive T-invariants we define the invariant weight of a transition $t_j \in T$ as $INV(V, t_j) = max_{v \in V}\{v_j\}$ where v_j denotes j-th component of v. This notation is extended to sets $S \subseteq T$ by $INV(V, S) = max_{t \in S}\{INV(V, t)\}$.

Another fact which might influence the size of the reachability set is the initial order of elements in the list. If we start with a list containing 77 $\{t_1, t_2\}$-elements followed by 21 $\{t_4\}$-elements and 33 $\{t_3\}$-elements we obtain 452 markings. The reason for this is that the enabling degree of t_1 and t_2 is greater than the enabling degree of the other transitions enabled at M_0, so that the firing of t_3 and t_4 is postponed more than necessary. To take this phenomenon partly into account we use the initialisation procedure of Fig. 11 to build up the list. (Obviously, others can be devised taking the enabling degree more gradually into account.) The enabling degree of a transition and for sets of transitions can be defined as follows:

Definition 14. The *enabling degree* of a transition $t \in T$ at marking M is defined as $ED(M, t) = max_{k \in \mathbb{N}_0}\{k | \forall p \in P : M(p) \geq kW(p, t)\}$ and its extension to sets $S \subseteq T$ is given by $ED(M, S) = min_{t \in S}\{ED(M, t)\}$.

The minimum is selected for sets here, since it determines how often a complete(!) set can be fired.

Let $Q_1, \ldots, Q_r$ be the equivalence classes of SCS^* for transitions of $T \setminus T^S_{low}$.
Define $Q := \bigcup_{i=1}^{r} Q_r$.
list_EC := empty
while $Q \neq \emptyset$ do
begin
    select $Q_i \in Q$ with $\forall j \in \{1, \ldots, r\} : ED(M_0, Q_j) \geq ED(M_0, Q_i)$
    for $k := 1$ to $INV(Q_i)$ do insert Q_i at the end of list_EC
    $Q := Q \setminus Q_i$;
end

Fig. 11. Initialisation procedure

The proposed algorithm should be a good alternative when the information provided by the T-invariants is not so trivial. This is e.g. the case for the weighted T-system presented in Fig. 12 [15]. The net models a painting line where several parts are processed at the same time. One interesting feature of this example (see [15]) is that at least 8 empty pallets are needed initially to make the system live. Fig. 13 shows some results for different initial markings. Not surprisingly, the exploitation of T-invariants leads to smaller reachability sets, since when the number of tokens increases all transitions can be fired exactly as often as specified by their T-invariant weight.

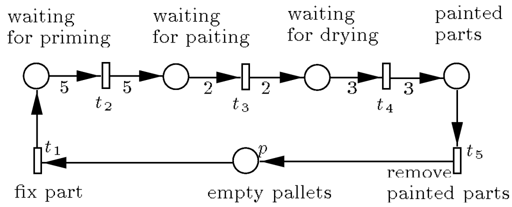

Fig. 12. A painting line

$M_0(p)$	$\|[M_0 >\|$	$\|[M_0 >_\rho\|$	stubborn
7	161	48	48
10	1001	91	107
20	10626	91	129
200	70058751	91	368
500	2.6566e+09	91	368

Fig. 13. Some results for the painting line with $V = \{(30, 6, 15, 10, 30)\}$.

Fig. 14 and 16 present two (extended) free choice nets and the corresponding results are given in Fig. 15 and 17 respectively. The net of Fig. 16 is taken from [6] and is a well-known example for a free choice net whose initial marking is not a home state. According to theorem 10(5) we have also defined $T_{low}^S = S := \{t, t'\}$ giving the potential to check whether the initial marking is a home state. These results are given in the column "$|[M_0 >_\rho> \mid$ home". Note that for the analysis of liveness only 7 markings need to be generated by setting $T_{low}^{\emptyset} = \emptyset$. Both tables show that the approach of considering T-invariants is more suitable when the marking of the net comprises several frozen tokens. Not surprisingly, finding deadlocks with the stubborn set method is more efficient, see $k = 3$ in Fig. 15.

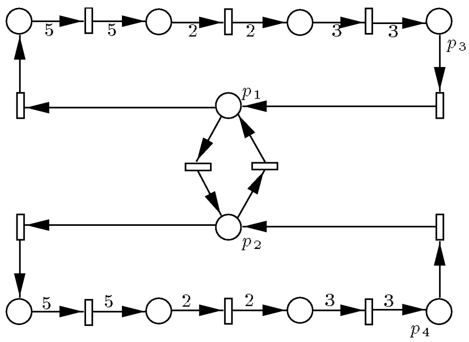

Fig. 14. A painting system

k	$\|[M_0 >_\rho\|$	stubborn
3	10523	9513
4	5468	11078
6	5472	11446
8	5476	11814
10	5480	12182
20	5500	14022
30	5520	15862
100	5660	28742

Fig. 15. Some results for the painting system with initial markings $M_0 = (k, k, k, k, 0, \ldots, 0)$.

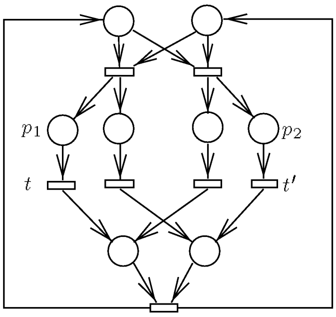

Fig. 16. A free choice net

k	$\vert[M_0 >_\rho\vert$	stubborn	$\vert[M_0 >_\rho \vert$ home
1	7	7	7
10	7	35	97
100	7	305	997
1000	7	3005	9997

Fig. 17. Results for initial markings $M_0 = (k, k, 0, \ldots, 0)$.

The proposed algorithm seems to perform better than the stubborn set method once the number of tokens is large which is e.g. the case in models for flexible manufacturing systems. Furthermore the dynamic priority method as presented in this article does not suffer from the ignoring problem and allows to check properties of specific markings. Not surprisingly the price for this has to be paid. Finally we consider a larger net from [5]. The net has been slightly modified in order to be live. We further defined T^S_{low} such that it can be checked whether the initial marking is a home state. The results given in Fig. 19 show that the proposed algorithm produces reachability sets of manageable size in comparison to the reachability set of the original Petri net.

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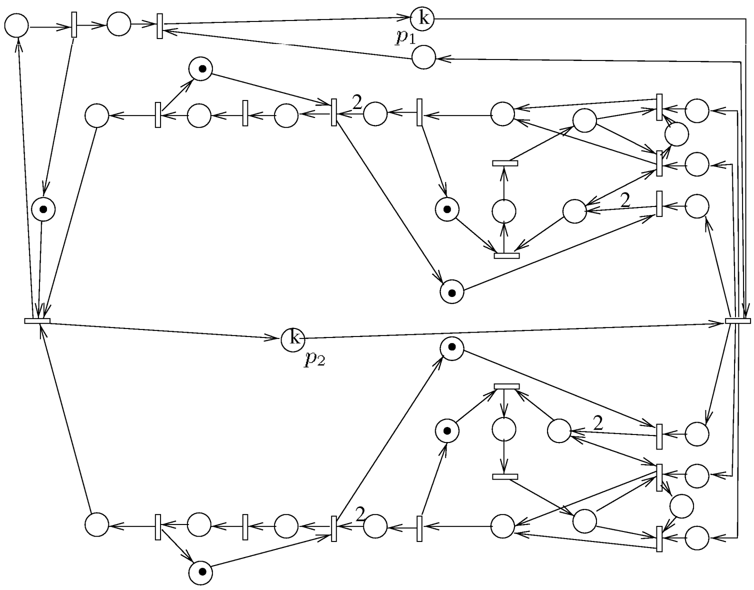

Fig. 18. The benchprod model

k	$\|[M_0 >\|$	$\|[M_0 >_\rho\|$ home	stubborn
1	172	100	28
2	2361	390	36
5	71560	2366	93
7	223894	4599	119
11	977842	11277	171
13	1665856	15723	197

Fig. 19. Some results for the benchprod model

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A Proof of lemma 7

Definition 15.

a) Difference of two firing sequences. $f, g \in T^*, t \in T$.

$$f - \varepsilon := f$$
$$f - t := \begin{cases} f & \text{if } t \notin f \\ f_1 f_2 & \text{if } f_1, f_2 \in T^* : t \notin f_1 \text{ and } f = f_1 t f_2 \end{cases}$$
$$f - tg := (f - t) - g$$

b) Permutation. The set $Perm(f)$ of all permutations of $f \in T^*$ is $Perm(f) := \{g \in T^* \mid (f - g)(g - f) = \varepsilon\}$.

Subtracting g from f cancels all transitions of g in f, provided they are part of f, where cancelling starts at the "beginning" of f. E.g. $t_2 t_1 t_1 t_3 t_5 t_4 - t_4 t_7 t_1 t_3 = t_2 t_1 t_5$. As usual concatenation has precedence on difference, i.e. $fg - h = (fg) - h$.

The next lemma states well-known properties of non-conflicting transitions concerning a reordering of firing sequences.

Lemma 16. *[1] Let N be a P/T-net and M denote some marking.*

a) $t, t' \in T : \bullet t \cap \bullet t' = \emptyset$. *Then* $M[t>, M[t'> \Longrightarrow M[tt'>$.
b) $t \in T, h \in T^* : \forall t' \in h : \bullet t \cap \bullet t' = \emptyset$. *Then* $M[ht>, M[t> \Longrightarrow M[th>$.
c) *Given* $f, \hat{g}, q \in T^*, h \in Perm(f - \hat{g})$ *where* $\forall t \in \hat{g}, t' \in q : \bullet t \cap \bullet t' = \emptyset$ *we have* $M[f>, M[hq> \Longrightarrow M[fq>$.

Proof of lemma 7:
If $g = \varepsilon$ the lemma trivially holds with $\tilde{g} := \varepsilon$.
So assume $g = t_1 \ldots t_k \neq \varepsilon$. $\tilde{g} = t_{i_1} \ldots t_{i_k}$ can be constructed as follows:

Set $\tilde{g} := \varepsilon; f := g$ and $r := 1$.
while $f \neq \varepsilon$ do
  if $(\exists t \in f : M[\tilde{g}t >_\rho)$ then
    choose $t_{i_r} \in f$ with $M[\tilde{g}t_{i_r} >_\rho$ such that
      $\exists f_1, f_2 \in T^* : f = f_1 t_{i_r} f_2$ and $\forall t \in f_1 : \neg M[\tilde{g}t >_\rho$ (*)
    Set $f := f - t_{i_r}; \tilde{g} := \tilde{g}t_{i_r}; r := r + 1$
  fi
od

The algorithm concatenates the leftmost ρ-enabled transition of f to the firing sequence $\tilde{g}$. The main part of the proof is to show the algorithm's termination. Define

$$s_r := min(\{1, \ldots, k\} \setminus \cup_{p=1}^{r-1}\{i_p\}), r = 1, \ldots, k$$

From this definition we directly get $s_{r+1} \geq s_r$ and

$$\cup_{j=1}^{s_r-1}\{j\} \subseteq \cup_{j=1}^{r-1}\{i_j\}$$

i.e. the set $\{1, \ldots, s_r - 1\}$ is a subset of $\{i_1, \ldots, i_{r-1}\}$.
With that the following property holds before each iteration of the while-loop:

$$f \neq \varepsilon \Longrightarrow M[\tilde{g}t_{s_r} > \qquad (4)$$

Proof by induction on r:

Base: $r = 1$

Since $s_1 = 1$, $M[t_1 >$ holds because of $M[g >$.

Step: $r \mapsto r + 1$

So assume $M[\tilde{g}t_{s_r} >$ holds. If the condition of the if-statement does not hold, then r is not incremented and we are done.
Now assume that the condition of the if-statement is satisfied. We have to show $M[\tilde{g}t_{i_r}t_{s_{r+1}} >$ where $\tilde{g}$ is the firing sequence in step r. We have $\bullet t' \cap \bullet t_{s_{r+1}} = \emptyset, \forall t' \in t_{i_1} \ldots t_{i_r} - t_1 \ldots t_{s_{r+1}-1}$ since otherwise $t_{s_{r+1}}$ would have been already chosen for firing, because of (*) and condition DEC. From construction we have $M[t_{i_1} \ldots t_{i_r} >$ and since $M[g >$ additionally $M[t_1 \ldots t_{s_{r+1}-1}t_{s_{r+1}} >$ holds.
Employing lemma 16c)[1] we get $M[\tilde{g}t_{i_r}t_{s_{r+1}} >$.

It remains to show that the while-loop terminates. Let us assume that it does not terminate, i.e. we reach a situation where $\forall t' \in g - \tilde{g} : \neg M[\tilde{g}t' >_\rho$. Because of (4) there is some enabled transition and thus there exists $t \notin g - \tilde{g} : M[\tilde{g}t >_\rho$ and condition DEC implies

$$\bullet t' \cap \bullet t = \emptyset, \forall t' \in g - \tilde{g} \qquad (5)$$

since otherwise t' would be ρ-enabled as well. Furthermore we have $t \notin T^S_{low}$, since otherwise (4) and condition DEC b) imply $\exists t' \in g - \tilde{g} : M[\tilde{g}t' >_\rho$ contradicting our assumption of a non-terminating while-loop. Employing lemma 16 again[2], (5) implies $M'[t >$ contradicting (3). So the while-loop terminates which accomplishes the proof. □

[1] Choose $f = t_{i_1} \ldots t_{i_r}, h = t_1 \ldots t_{s_{r+1}-1}, \hat{g} = t_{i_1} \ldots t_{i_r} - t_1 \ldots t_{s_{r+1}-1}, q = t_{s_{r+1}}$.
[2] Choose $f = g, \hat{g} = g - \tilde{g}$ and $q = t$. Note $g - (g - \tilde{g}) = \tilde{g}$ since $t \in \tilde{g} \Rightarrow t \in g$.

The following part was NOT included in the proceedings of the 18th International Conference on Application and Theory of Petri Nets, Toulouse (France), June 1997.

Considering only liveness issues one can modify condition DEC as follows giving a better reduction on the reachability graph size. The idea is inspired by the stubborn set method.

Definition 17 modified condition DEC (MDEC).
A priority P/T-net satisfies the modified condition *dynamically EQUAL-Conflict (MDEC)* iff

$$\forall M \in \mathbb{N}_0^{|P|} : \forall t \in T : M[t >_\rho \implies \begin{array}{l} \text{a) } \forall t' \in [t]_{SCS^*} : M[t' >_\rho \quad \underline{\text{or}} \\ \text{b) } t \in \Omega_M \implies \Omega_M = T^S_{low}\,. \end{array}$$

What has changed is that now also transitions belonging to T^S_{low} might be of higher priority than all other transitions provided that all transitions in structural conflict, i.e. all transitions of $[t]_{SCS^*}$, are enabled as well. Unfortunately lemma 7 does not hold in this case, so that we are not able to consider specific markings of the net.

Fig. 20 shows a counterexample. Defining ρ such that $\{t_1, t_2\}$ has priority on $\{t_3, t_4\}$ at M_0 shows that the reachable marking $(2,0)$ is not ρ-reachable, i.e. $(2,0) \notin [M_0 >_\rho$, although it satisfies (3) with $T^S_{low} := T$.

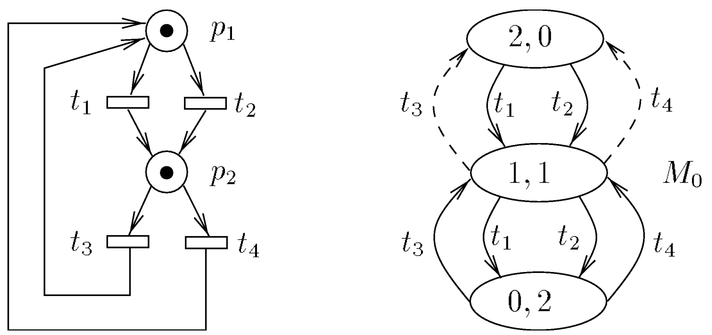

Fig. 20. Counterexample for ρ-reachability of specific markings in the context of condition MDEC

First we show that no transition is ignored, i.e. ρ-traps do not exist. The proof uses the same arguments as before.

Lemma 18. *Let N be strongly connected and (N, M_0) be bounded. If (N, ρ) satisfies condition MDEC, then (N, ρ, M_0) has no ρ-trap.*

Proof. Assume (N, ρ, M_0) has a ρ-trap. Since (N, M_0) is bounded, $[M' >_\rho$ is finite $\forall M' \in [M_0 >$ (cf. lemma 6), thus $[M >_\rho$ contains a final strongly connected component $C_M \subseteq [M >_\rho$. Define $\tilde{T} := \{t \in T | \exists M' \in C_M : M'[t >_\rho\}$ and $\tilde{P} := \bullet\tilde{T}$. $\tilde{T}$ can be regarded as the set of ρ-live transitions with respect to C_M. Since N is strongly connected and because of our assumption of the existence of a ρ-trap implying $\tilde{T} \neq \emptyset$ and $(T \setminus \tilde{T}) \neq \emptyset$ (note that in this case also $T_{NEC} \subseteq T^S_{low} \subseteq T \setminus \tilde{T}$ and $T_{EC} \subseteq \tilde{T}$), one of the following two cases holds (cf. Fig. 21):

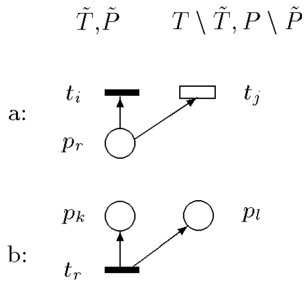

Fig. 21. Illustration of proof

a: $\exists p_r \in \tilde{P} : p_r\bullet \supseteq \{t_i, t_j\}$ and $t_i \in \tilde{T}, t_j \in T \setminus \tilde{T}$.
Since condition MDEC holds, part a) implies that if $M'[t_i >_\rho$ for some $M' \in C_M$, then $M'[t_j >_\rho$, contradicting $t_j \notin \tilde{T}$.

b: $\exists t_r \in \tilde{T} : t_r\bullet \supseteq \{p_k, p_l\}$ and $p_k \in \tilde{P}, p_l \in P \setminus \tilde{P}$.
Since $t_r \in \tilde{T}$, t_r might fire infinitely often with regard to C_M. Since $p_l \notin \tilde{P} = \bullet\tilde{T}$, place p_l is not bounded, contradicting boundedness of (N, M_0).

Thus assuming the existence of a ρ-trap was wrong which completes the proof. □

The lemma shows that not all transitions of T^S_{low} can be ignored. Furthermore the arguments of the proof actually show that no transition can be ignored provided it is enabled at some marking of the set C_M. In that case the set $T \setminus \tilde{T}$ denotes the set of ignored transitions giving us the same contradictions, because of boundedness and strongly connectedness of (N, M_0).

As shown by the net of Fig. 20, lemma 7 does not hold, but in the context of condition MDEC we can prove a similar statement in the context of strongly connected and bounded nets. Since the sets $EN_\rho(M)$ for priorities satisfying condition MDEC are also special cases of stubborn sets the following lemma states the same as theorem 1.29 of [19].

Lemma 19. *Let N be strongly connected, (N, M_0) be bounded and (N, ρ) satisfy condition MDEC. Let $M \in [M_0 >$ and $g \in T^*$ be a firing sequence such that $M[g >$. Then*
$\exists h \in T^* : \exists \tilde{g} \in Perm(gh) : M[\tilde{g} >_\rho$

I.e. one can always find an extension of a firing sequence such that a permutation of this extension has concession in (N, ρ) as well.

Proof. If $g = \varepsilon$ the lemma trivially holds with $\tilde{g} := \varepsilon$.

So assume $g = t_1 \dots t_k \neq \varepsilon$. $\tilde{g} = t_{i_1} \dots t_{i_k}$ can be constructed as follows:

```
Set g̃ := ε; f := g and r := 1.
while f ≠ ε do
    if (∃t ∈ f : M[g̃t >_ρ) then
        choose t_{i_r} ∈ f with M[g̃t_{i_r} >_ρ such that
            ∃f_1, f_2 ∈ T* : f = f_1 t_{i_r} f_2 and ∀t ∈ f_1 : ¬M[g̃t >_ρ
          Set f := f − t_{i_r}; g̃ := g̃t_{i_r}; r := r + 1
    else
        choose an arbitrary t ∈ T* with M[g̃t >_ρ and
          set g̃ := g̃t
    fi
od
```

Analogous to the proof of lemma 7 we have for

$$s_r := min(\{1, \dots, k\} \setminus \cup_{p=1}^{r-1}\{i_p\}), r = 1, \dots, k$$

that the following property holds before each iteration of the while-loop:

$$f \neq \varepsilon \Longrightarrow M[\tilde{g}t_{s_r} > \qquad (6)$$

With that the termination of the while-loop is directly implied, since $EN(M) \neq \emptyset$ guarantees $EN_\rho(M) \neq \emptyset$ and ignorance does not occur due to lemma 18. Thus $f = \varepsilon$ will eventually hold. (Note that $[M_0 >$ is finite.) □

Thanks to lemma 18 ignorance does not occur, so that the following holds using the same arguments as given in [19].

Theorem 20. *Let N be strongly connected, (N, ρ) satisfying condition MDEC and (N, M_0) be bounded. Then the following holds:*

1. (N, M_0) *dead* $\Longleftrightarrow$ (N, ρ, M_0) *ρ-dead*
2. (N, M_0) *live* $\Longleftrightarrow$ (N, ρ, M_0) *ρ-live*

The application of condition MDEC for our approach leads to the following algorithms, where now the SCS^*-sets of all(!) transitions are considered.

With these slightly modified algorithms liveness can be examined on a usually smaller reachability set of the priority P/T-net with a general net structure.

Fig. 24 shows the corresponding results for the benchprod model (cf. Figs. 18 and 19).

Calculate $EN_\rho(M)$ in each generation step of the reachability analysis as follows:
Let $Q_1, \ldots, Q_r$ be the equivalence classes of SCS^* for transitions of T.
Define $\hat{Q} := \bigcup_{i=1}^{r} \{Q_i | Q_i \subseteq EN(M)\}$
if $\hat{Q} = \emptyset$ return $EN(M)$
else
begin
    find first element e in list_ALL with $e \in \hat{Q}$
    remove e from list_ALL and insert it at the end of list_ALL
    return e
end

Fig. 22. A modified algorithm for determining $EN_\rho(M)$.

Let $Q_1, \ldots, Q_r$ be the equivalence classes of SCS^* for transitions of T.
Define $Q := \bigcup_{i=1}^{r} Q_r$.
list_ALL := empty
while $Q \neq \emptyset$ do
begin
    select $Q_i \in Q$ with $\forall j \in \{1, \ldots, r\} : ED(M_0, Q_j) \geq ED(M_0, Q_i)$
    for $k := 1$ to $INV(Q_i)$ do insert Q_i at the end of list_ALL
    $Q := Q \setminus Q_i$;
end

Fig. 23. Modified initialisation procedure

k	$\|[M_0 >\|$	$\|[M_0 >_\rho\|$	stubborn
1	172	100	28
2	2361	298	36
5	71560	877	93
7	223894	1263	119
11	977842	2035	171
13	1665856	2421	197

Fig. 24. Some results for the benchprod model with priorities satisfying the condition MDEC using the modified algorithm