Fitting Markovian Arrival Processes by Incorporating Correlation into Phase Type Renewal Processes

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3. Fitting Lag-1 joint moments $E[X_0^i X_1^j]$
4. Fitting Higher Lag Joint Moments $E[X_0^i X_\ell^j], \ell \geq 1$
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Introduction

Advantages of modeling inter-arrival/service times by
Phase-type (PH) distributions and Markovian Arrival Processes (MAPs)

- Markov property
- efficient numerical methods for QN models (MAP/PH/1, ...)
- easy to simulate

Fitting methods:

- EM based methods (use trace directly; applicable to "small" traces)
- matching based methods (fit statistical figures of a trace;
  applicable to larger traces)

Experience from distribution fitting:
Methods operating on special sub-classes of PH distributions are more effective (Acyclic PH, Hyper-Erlang,...)

Introduction (cont'd)

Recent matching methods for MAPs follow a "two-phase" approach:

1. fit distribution (giving a PH distribution)
2. fit correlation

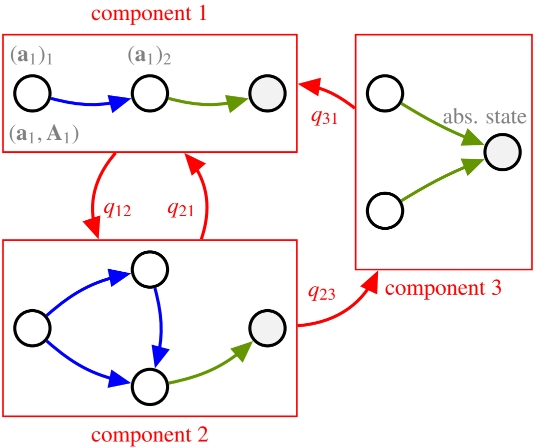
Which figures to fit?

Telek, Horváth (2007):
A (non-redundant) MAP with n states is characterized by n^2 moments
($2n-1$ marginal moments, $(n-1)^2$ lag-1 joint moments $E[X_0^i X_1^j]$)

This paper

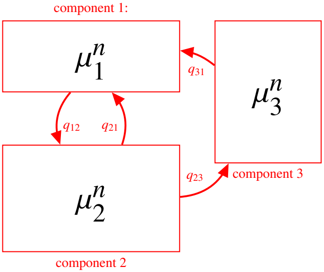
- Assumption: PH distribution representing inter-arrival times is given
- Fitting is based on joint moments $E[X_0^i X_\ell^j]$
- Sub-class of MAPs (Structured MAPs (SMAPs)) is considered

Structured Markovian Arrival Process (SMAP)



- set of component (PH) distributions $\{(\mathbf{a}_i, \mathbf{A}_i) \mid i = 1, \ldots, N\}$
- discrete time Markov chain, specifying which component generates next inter-arrival time; irreducible switching matrix $\mathbf{Q} = (q_{ij})$

Structured Markovian Arrival Process (cont'd)



$(\{(\mathbf{a}_i, \mathbf{A}_i) \mid i = 1, \ldots, N\}, \mathbf{Q})$:

$$E[X^n] = \sum_{i=1}^{N} \pi_i \mu_i^n \quad (1)$$

$$\eta^{(1)}_{n_1 n_2} = E[X_0^{n_1} X_1^{n_2}] = \sum_{i=1}^{N} \sum_{j=1}^{N} \pi_i \, q_{ij} \, \mu_i^{n_1} \mu_j^{n_2} \quad (2)$$

$\mu_i^n := n! \, \mathbf{a}_i (-\mathbf{A}_i)^{-n} \mathbf{1}^T$	n-th moment of distribution of component i
$\pi \mathbf{Q} = \pi, \pi \mathbf{1}^T = 1$	steady-state distribution of "switching process"

- Sizes of Eqs. (1) and (2) only depend on N

How to get Component Distributions $\{(\mathbf{a}_i, \mathbf{A}_i) \mid i = 1, \ldots, N\}$?

Assume PH distribution $(\boldsymbol{\pi}, \mathbf{H})$ for inter-arrival times is given, e.g. from moment matching or EM-based fitting methods (we used: G-FIT).

Define

$$\begin{aligned} N &:= \#\text{ non-zero elements in } \boldsymbol{\pi} \\ \mathbf{A}_i &:= \mathbf{H} \\ \mathbf{a}_i &:= \begin{cases} \mathbf{e}_i & \text{if } \pi_i > 0 \\ \mathbf{0} & \text{otherwise} \end{cases} \end{aligned}$$

<u>Constraint for switching matrix $\mathbf{Q}$:</u> $\boldsymbol{\pi}\mathbf{Q} = \boldsymbol{\pi}$ since

$$\sum_{i=1}^{N} \pi_i \mathbf{a}_i e^{\mathbf{A}_i t} \mathbf{1}^T = \sum_{i=1}^{N} \pi_i \mathbf{e}_i e^{\mathbf{H} t} \mathbf{1}^T = \boldsymbol{\pi} e^{\mathbf{H} t} \mathbf{1}^T \quad \text{for } t \geq 0$$

i.e. distribution $(\boldsymbol{\pi}, \mathbf{H})$ remains unchanged.

How to get Component Distributions? (cont'd)

Furthermore one can apply similarity transformations for more suitable representation of PH-dist $(\boldsymbol{\pi}, \mathbf{H})$

$$\boldsymbol{\pi}' = \boldsymbol{\pi}\mathbf{B}, \mathbf{H}' = \mathbf{B}^{-1}\mathbf{H}\mathbf{B}$$

Elementary transformation matrix

$$\mathbf{B}_{i,j}(b) = \begin{array}{c} \\ \\ \\ i \\ \\ \\ \end{array} \begin{bmatrix} 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & \dots \\ \vdots & \vdots & \ddots & \ddots & \dots \\ 0 & b & 0 & 1-b & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$
$$\qquad\qquad\qquad j$$

Fitting Lag-1 joint moments

System of lin. Eqs. for $\mathbf{Q}$: (with constraints $\boldsymbol{\pi}\mathbf{Q} = \boldsymbol{\pi}$, $\mathbf{Q}\mathbf{1}^T = \mathbf{1}$, $\mathbf{Q} \geq \mathbf{0}$)

$$\eta^{(1)}_{n_1 n_2} = E[X_0^{n_1} X_1^{n_2}] = \sum_{i=1}^{N} \sum_{j=1}^{N} \boldsymbol{\pi}_i \, q_{ij} \, \mu_i^{n_1} \mu_j^{n_2}, \quad n_1, n_2 = 1, \ldots, K$$

should be close to given lag-1 joint moments $\zeta^{(1)}_{n_1 n_2}$ from trace

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NNLS (non-negative least-squares problem)

Given matrices $\mathbf{A}, \mathbf{B}$ and vectors $\mathbf{a}, \mathbf{b}$.
Determine solution $\mathbf{x}$ minimizing $||\mathbf{A}\mathbf{x} - \mathbf{a}||_2^2$ subject to $\mathbf{B}\mathbf{x} = \mathbf{b}, \mathbf{x} \geq 0$

- Well-known problem (e.g. *Lawson/Hanson: Solving Least Squares Problems, Prentice-Hall, 1974*)
- Efficient algorithms are reported in the literature (can handle several hundreds/thousands of variables/constraints)
- Implementations are available, e.g. R (limSolve), MATLAB (lsqlin, which we used).

Fitting Lag-1 joint moments (cont'd)

Problem: Higher order joint moments dominate optimization.

Possible Solutions:

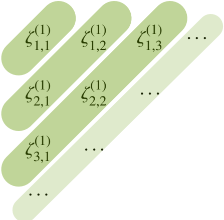
- weighting of joint moments, but no general rule which weight functions are appropriate

Here:

- relative errors; NNLS: $\sum_{n_1,n_2}\left(\frac{\sum_{i,j}\pi_i\mu_i^{n_1}\mu_j^{n_2}q_{ij}-\zeta^{(1)}_{n_1,n_2}}{\zeta^{(1)}_{n_1,n_2}}\right)^2$
- step-by-step

  solve NNLS for subset of
  $S := \{\zeta^{(1)}_{n_1 n_2} \mid n_1, n_2 = 1, \ldots, K\}$
  encode resultant solutions $\eta^{(1)}_{n_1 n_2}$ as linear constraints for next NNLS problem

  (*"one-step, if subset=S"*)



Fitting Higher Lag Joint Moments

Idea: incorporate memory into switching process, i.e. determine
$q^{(k)}_{(i_k|i_0\ldots i_{k-1})} := P[\text{component } i_k \mid \text{components } i_0 \ldots i_{k-1}]$ which

- approximates lag-k joint moments and
- keeps fitting results for lag-ℓ joint moments, $\ell < k$

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$$r^{(k)}_{i_0\ldots i_{k-1}} := P[\text{comps } i_0 \ldots i_{k-1}] = \pi_{i_0} q^{(1)}_{(i_1|i_0)} q^{(2)}_{(i_2|i_0 i_1)} \cdots q^{(k-1)}_{(i_{k-1}|i_0\ldots i_{k-2})}$$

$$E[X_0^{n_1} X_k^{n_2}] = \sum_{i_0,\ldots,i_k=1}^{N} \mu_{i_0}^{n_1} \mu_{i_k}^{n_2} r^{(k)}_{i_0\ldots i_{k-1}} q^{(k)}_{(i_k|i_0\ldots i_{k-1})}$$

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Constraints: $r^{(k)}_{i_1\ldots i_k} = \sum_{i_0=1}^{N} r^{(k)}_{i_0\ldots i_{k-1}} q^{(k)}_{(i_k|i_0\ldots i_{k-1})}$ $\qquad \sum_{i_k=1}^{N} q^{(k)}_{(i_k|i_0\ldots i_{k-1})} = 1$

relative error; NNLS: $\sum_{n1,n2} \left(\frac{\sum_{i_0,\ldots,i_k=1}^{N} \mu_{i_0}^{n_1} \mu_{i_k}^{n_2} r^{(k)}_{i_0\ldots i_{k-1}} q^{(k)}_{(i_k|i_0\ldots i_{k-1})} - \zeta^{(k)}_{n_1,n_2}}{\zeta^{(k)}_{n_1,n_2}} \right)^2$

Fitting Higher Lag Joint Moments (cont'd)

From $q^{(k)}_{(i_k|i_0 \dots i_{k-1})}$ an SMAP $((\{(\mathbf{a}_i, \mathbf{A}_i) \mid i = 1, \dots, N^k\}, \mathbf{Q}))$

can be obtained by encoding the memory of the switching process into the components

$$q_{a,b} = \begin{cases} q^{(k)}_{(i_k|i_0 \dots i_{k-1})} & \text{if } a = (i_0, \dots, i_{k-1}) \text{ and } b = (i_1, \dots, i_k) \\ 0 & \text{otherwise} \end{cases}$$

PH distribution of component $(i_1, \dots, i_k)$ is given by $(\mathbf{a}_{i_k}, \mathbf{A}_{i_k})$.

Fitting of SMAPs up to lag *k* *("basic idea")*

taking similarity transformations into account
(experience: improves fitting results)

Given PH distribution $(\boldsymbol{\pi}, \mathbf{H})$
for several similarity transformations **do**
- generate component distributions
- fit up to lag k *(either step-by-step or in one-step)*

end for
return best result $(\boldsymbol{\pi}_{best}, \mathbf{H}_{best}, \mathbf{Q}_{best})$

Application Examples

1. Fitting of a given MAP:

$$\mathbf{D}_0=\begin{bmatrix} -3.721 & 0.5 & 0.02 \\ 0.1 & -1.206 & 0.005 \\ 0.001 & 0.002 & -0.031 \end{bmatrix},\ \mathbf{D}_1=\begin{bmatrix} 0.2 & 3.0 & 0.001 \\ 1.0 & 0.1 & 0.001 \\ 0.005 & 0.003 & 0.02 \end{bmatrix},\ \boldsymbol{\pi} = \boldsymbol{\pi}\left(-\mathbf{D}_0\right)^{-1}\mathbf{D}_1,\ \boldsymbol{\pi}\mathbf{1}^T = 1$$

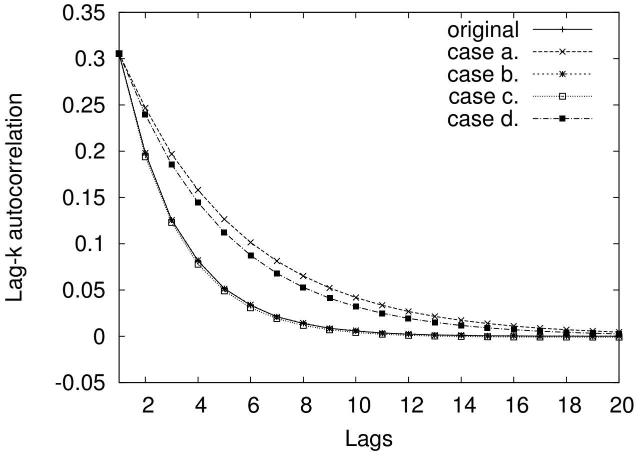
	Method	Subject of fitting	# states of MAP
case a.	step-by-step	$\eta_{i,j}^{(k)},\quad i,j=\{1,2\}, k=1$	9
case b.	step-by-step	$\eta_{i,j}^{(k)},\quad i,j=\{1,2\}, k=\{1,2\}$	27
case c.	step-by-step	$\eta_{i,j}^{(k)},\quad i,j=1, k=\{1,2,3\}$	81
case d.	one-step	$\eta_{i,j}^{(k)},\quad i,j=\{1,2\}, k=1$	9

a and b consider higher order joint moments

c considers only order 1 joint moments, but higher lags

d =a, but one-step

Results of Fitting a given MAP



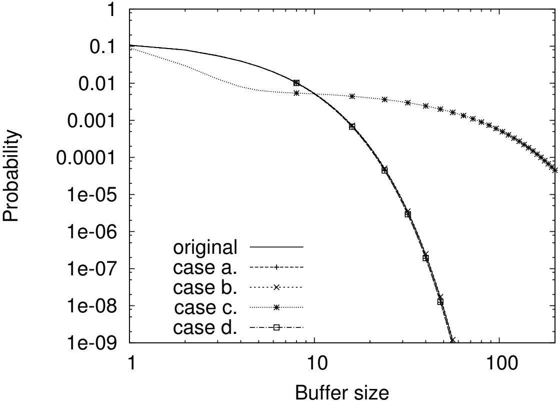
lag-k autocorrelation:

$$\rho_k = \frac{(\eta_{11}^{(k)} - E[X]^2)}{(E[X^2] - E[X]^2)}$$

- cases b,c give better fitting than a,d (since they consider higher lags)
- no difference between step-by-step and one-step

Results of Fitting a given MAP (cont'd)

MAP/M/1 queue length distribution (low load $\rho = 0.38$)



- cases a,b,d (considering higher order lag-1 joint moments) give better results than case c
- similar results for higher loads

Conforms to theory: "lag-1 joint moments determine MAP"

Application Examples (cont'd)

2. Fitting of LBL-TCP-3 trace

Distribution fitted with G-FIT giving a Hyper-Erlang distribution
(4 branches, each with two-phase Erlang distribution)

	Method	Subject of fitting	Time (sec.)	# states of MAP
case a.	step-by-step	$\eta_{i,j}^{(k)}, \quad i,j = \{1,2,3\}, k = 1$	≈ 2	16
case b.	step-by-step	$\eta_{i,j}^{(k)}, \quad i,j = \{1,2\}, k = \{1,2\}$	≈ 5	44
case c.	step-by-step	$\eta_{i,j}^{(k)}, \quad i,j = 1, k = \{1,2,3\}$	≈ 20	176
case d.	one-step	$\eta_{i,j}^{(k)}, \quad i,j = \{1,2,3\}, k = 1$	≈ 2	11
case e.	one-step	$\eta_{i,j}^{(k)}, \quad i,j = \{1,2\}, k = \{1,2\}$	≈ 5	32

(Intel Quad Core, 2.8GHz)

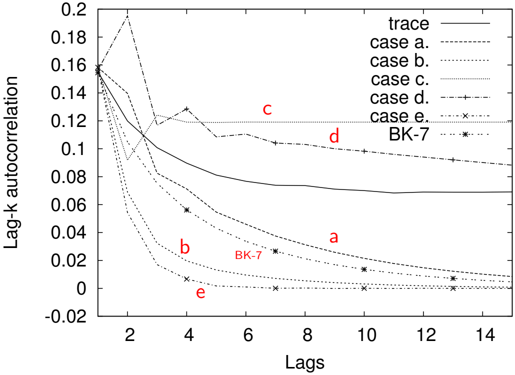
a,b,c decreasing order of joint moments, increasing lags
d=a, e=b but one-step

For comparison best MAP of

P. Buchholz, J. Kriege. *A Heuristic Approach for Fitting MAPs to Moments and Joint Moments.* QEST 2009.

was used: "BK-7" with 7 states.

Results of Fitting LBL Trace

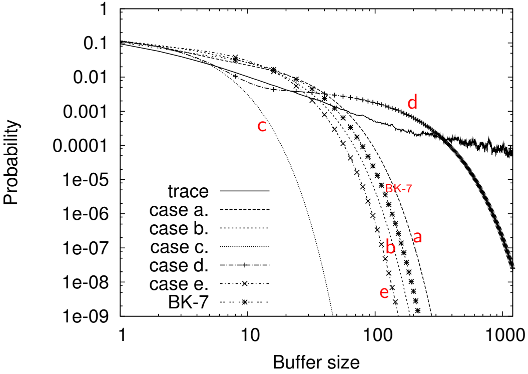


a,b,c
(decr. order of joint moments, incr. lags)
d=a, e=b (but one-step)

- fitting based on lag-1 joint moments gave better results

Results of Fitting LBL Trace (cont'd)

MAP/M/1 queue length distribution (high load $\rho = 0.8$)



a,b,c
(decr. order of joint moments, incr. lags)
d=a, e=b (but one-step)

- higher order lag-1 joint moments seem to be "more important" than higher lags

Conclusions

- Presented method allows incorporation of correlation into PH-type distributions
  - utilizing similarity transformations
- Special structure of SMAPs makes it possible to formulate joint moment fitting as an NNLS problem
  - also considering higher lags
  - using relative errors
  - with fitting step-by-step or in one-step
    (LBL: different results concerning rel. errors of joint moments, but showed no relevant improvement concerning acf and QN results)
  - complexity depends on the number of components
- resultant MAPs might get large; no problem in our experiments (at least usable for simulation)

Future work:

Systematically increase number of components for better fitting