An Improved Method for Bounding Stationary Measures of Finite Markov Processes

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- Basic Assumptions
- Bounding Approach by Courtois/Semal
- Improved Bounds
- A new Bounding Algorithm
- Conclusions and Outlook

Basic Definitions

We consider Markov Chains in discrete time with

- State space $\mathbb{S} = \{1,\dots, n\}$
- Reward vector $\mathbf{r} \in \mathbb{R}_+^n$
- A lower bound on the transition probabilities $\mathbf{L} \in \mathbb{R}^{n\times n}$ with
  $\mathbf{L} \geq \mathbf{0}$ and $\mathbf{L}\mathbf{e}^T \leq 1$
- An upper bound on the transition probabilities $\mathbf{U} \in \mathbb{R}^{n\times n}$ with
  $\mathbf{U} \geq \mathbf{L}$ and $\mathbf{U}\mathbf{e}^T \geq 1$

- Goal: Computation of bounds for $\mathbf{pr}$
  where $\mathbf{pP}=\mathbf{p}$, $\mathbf{pe}^T=1$ and
  $\mathbf{P}$ is an irreducible stochastic matrix with $\mathbf{L} \leq \mathbf{P} \leq \mathbf{U}$

Basic Definitions

Reasons for knowing only bounds on transition probabilities:

- Limited knowledge of the application and its environment
- Limited accuracy of measurements
- Abstraction to achieve Markovian behavior
- Markov chains resulting from decomposition and aggregation

Approach can be extended to continuous time Markov chains with

- Bounds on the transition rate matrix $\mathbf{V} \leq \mathbf{W}$ such that
  - $\mathbf{V}(i,j)$, $\mathbf{W}(i,j) \geq 0$ for $i \neq j$ and $\mathbf{V}(i,i) \leq -\Sigma_{j \neq i} \mathbf{V}(i,j)$, $\mathbf{W}(i,i) \leq -\Sigma_{j \neq i} \mathbf{W}(i,j)$

by using randomization
(i.e., $\mathbf{L} = \mathbf{V}/\alpha + \mathbf{I}$ and $\mathbf{U} = \mathbf{W}/\alpha + \mathbf{I}$, $\alpha \geq \max(|\mathbf{V}(i,i)|)$)

Basic Definitions

Definition of sets of matrices

- $\mathbb{P}_{\mathbf{L}} = \{\mathbf{P} \mid \mathbf{P} \geq \mathbf{L}, \mathbf{P}\mathbf{e}^T = \mathbf{e}^T \text{ and } \mathbf{P} \text{ irreducible}\}$
  - $\mathbb{V}_{\mathbf{L}} = \{\mathbf{v} \mid \mathbf{v} \geq \mathbf{0}, \exists \mathbf{P} \in \mathbb{P}_{\mathbf{L}}: \mathbf{v}\mathbf{P} = \mathbf{v} \text{ and } \mathbf{v}\mathbf{e}^T = 1\}$
- $\mathbb{P}_{\mathbf{U}} = \{\mathbf{P} \mid \mathbf{P} \leq \mathbf{U}, \mathbf{P}\mathbf{e}^T = \mathbf{e}^T \text{ and } \mathbf{P} \text{ irreducible}\}$
  - $\mathbb{V}_{\mathbf{U}} = \{\mathbf{v} \mid \mathbf{v} \geq \mathbf{0}, \exists \mathbf{P} \in \mathbb{P}_{\mathbf{U}}: \mathbf{v}\mathbf{P} = \mathbf{v} \text{ and } \mathbf{v}\mathbf{e}^T = 1\}$
- $\mathbb{P}_{\mathbf{L},\mathbf{U}} = \mathbb{P}_{\mathbf{L}} \cap \mathbb{P}_{\mathbf{U}}$
  - $\mathbb{V}_{\mathbf{L},\mathbf{U}} = \{\mathbf{v} \mid \mathbf{v} \geq \mathbf{0}, \exists \mathbf{P} \in \mathbb{P}_{\mathbf{L},\mathbf{U}}: \mathbf{v}\mathbf{P} = \mathbf{v} \text{ and } \mathbf{v}\mathbf{e}^T = 1\}$

Find bounds $\min_{\mathbf{v} \in \mathbb{V}_{\mathbf{L},\mathbf{U}}} (\mathbf{v}\mathbf{r})$ and $\max_{\mathbf{v} \in \mathbb{V}_{\mathbf{L},\mathbf{U}}} (\mathbf{v}\mathbf{r})$

- Bounds on the expected reward knowing bounds on transition probabilities/rates

Courtois/Semal Bounds

Bounds based on $\mathbb{P}_{\mathbf{L}}$ (Courtois/Semal JACM 1984)

- Let $\mathbf{N} = (\mathbf{I} - \mathbf{L})^{-1}$ and $\mathbf{Z} = (\text{diag}(\mathbf{N}\mathbf{e}^T))^{-1}\mathbf{N}$
  - $\mathbf{N} \geq 0$ exists and $\mathbf{Z}$ is a stochastic matrix
- $\min_{\mathbf{v} \in \mathbb{V}_{\mathbf{L}}} (\mathbf{vr}) = \min_{i \in \{1,\dots,n\}} (\mathbf{e}_i\mathbf{Z}\mathbf{r})$ and
  $\max_{\mathbf{v} \in \mathbb{V}_{\mathbf{L}}} (\mathbf{vr}) = \max_{i \in \{1,\dots,n\}} (\mathbf{e}_i\mathbf{Z}\mathbf{r})$

Bounds based on $\mathbb{P}_{\mathbf{U}}$ (Courtois/Semal JACM 1984)

- Let $\mathbf{M} = (\mathbf{I} - \mathbf{U})^{-1}$ and $\mathbf{Y} = (\text{diag}(\mathbf{M}\mathbf{e}^T))^{-1}\mathbf{M}$
  - If $\mathbf{M}$ exists and $\mathbf{M}\mathbf{e}^T < 0$, $\mathbf{eM} < 0$, then $\mathbf{Y}$ is a stochastic matrix and
  - $\min_{\mathbf{v} \in \mathbb{V}_{\mathbf{U}}} (\mathbf{vr}) = \min_{i \in \{1,\dots,n\}} (\mathbf{e}_i\mathbf{Y}\mathbf{r})$ and
    $\max_{\mathbf{v} \in \mathbb{V}_{\mathbf{U}}} (\mathbf{vr}) = \max_{i \in \{1,\dots,n\}} (\mathbf{e}_i\mathbf{Y}\mathbf{r})$

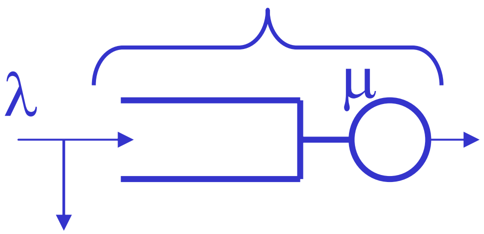
Courtois/Semal Bounds

- Best possible bounds if only $\mathbb{P}_L$ or $\mathbb{P}_U$ are known (but not the best bounds if $\mathbb{P}_{L,U}$ is known!)
- Extensions and improvements of the approach until recently
  - Better bounds and alternative bound computation if the method is used in combination with aggregation e.g. Franceschinis/Muntz 1984, Semal 1995
  - Improved bounds, partially based on $\mathbb{P}_{L,U}$ for upper Hessenberg matrices e.g. Muntz/Lui 1994, Mahevas/Rubino 2001

A new approach for bound computation based on $\mathbb{P}_{L,U}$ that is applicable for arbitrary matrices

Courtois/Semal Bounds

An example to clarify the problem:
Mean population/blocking prob. in a M/M/1/K system with $\lambda \in [\lambda^-, \lambda^- + \varepsilon_1]$ and $\mu \in [\mu^-, \mu^- + \varepsilon_2]$
Results can be computed only from **L** not from **U** for our parameters!



Lower bounds of transition rates
(diagonal elements are not printed)

$$\begin{pmatrix} -\Sigma & \lambda^- & & & \\ \mu^- & -\Sigma & \lambda^- & & \\ & \ddots & \ddots & \ddots & \\ & & \mu^- & -\Sigma & \lambda^- \\ & & & \mu^- & -\Sigma \end{pmatrix}$$

Courtois/Semal Bounds

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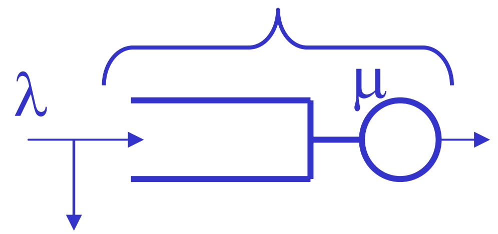
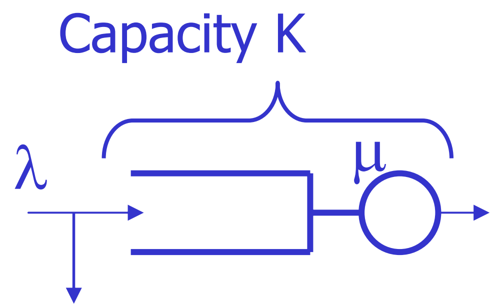


Matrix yielding the lower bound for the rewards
(diagonal elements are not printed)

$$\begin{pmatrix} -\Sigma & \lambda^- & & & \\ \mu^- + \varepsilon_1 + \varepsilon_2 & -\Sigma & \lambda^- & & \\ \vdots & \ddots & \ddots & \ddots & \\ \varepsilon_1 + \varepsilon_2 & & \mu^- & -\Sigma & \lambda^- \\ \varepsilon_1 + \varepsilon_2 & & & \mu^- & -\Sigma \end{pmatrix}$$

Courtois/Semal Bounds



An example to clarify the problem:
Mean population/blocking prob. in a M/M/1/K system with $\lambda \in [\lambda^-, \lambda^- + \varepsilon_1]$ and $\mu \in [\mu^-, \mu^- + \varepsilon_2]$
Results can be computed only from **L** not from **U** for our parameters!

Matrix yielding the upper bound for the rewards
(diagonal elements are not printed)

$$
\begin{pmatrix}
-\Sigma & \lambda^- & & & \varepsilon_1 + \varepsilon_2 \\
\mu^- & -\Sigma & \lambda^- & & \varepsilon_1 + \varepsilon_2 \\
 & \ddots & \ddots & \ddots & \vdots \\
 & & \mu^- & -\Sigma & \lambda^- + \varepsilon_1 + \varepsilon_2 \\
 & & & \mu^- & -\Sigma
\end{pmatrix}
$$

Matrices for the bounds include transitions changing the population in the system by adding/removing K customers in one step $\Rightarrow$ impossible from the system specification!!

Improved Bounds

Matrices yielding better bounds (in this case easy to prove)

Matrix yielding the lower bound for the rewards (diagonal elements are not printed)

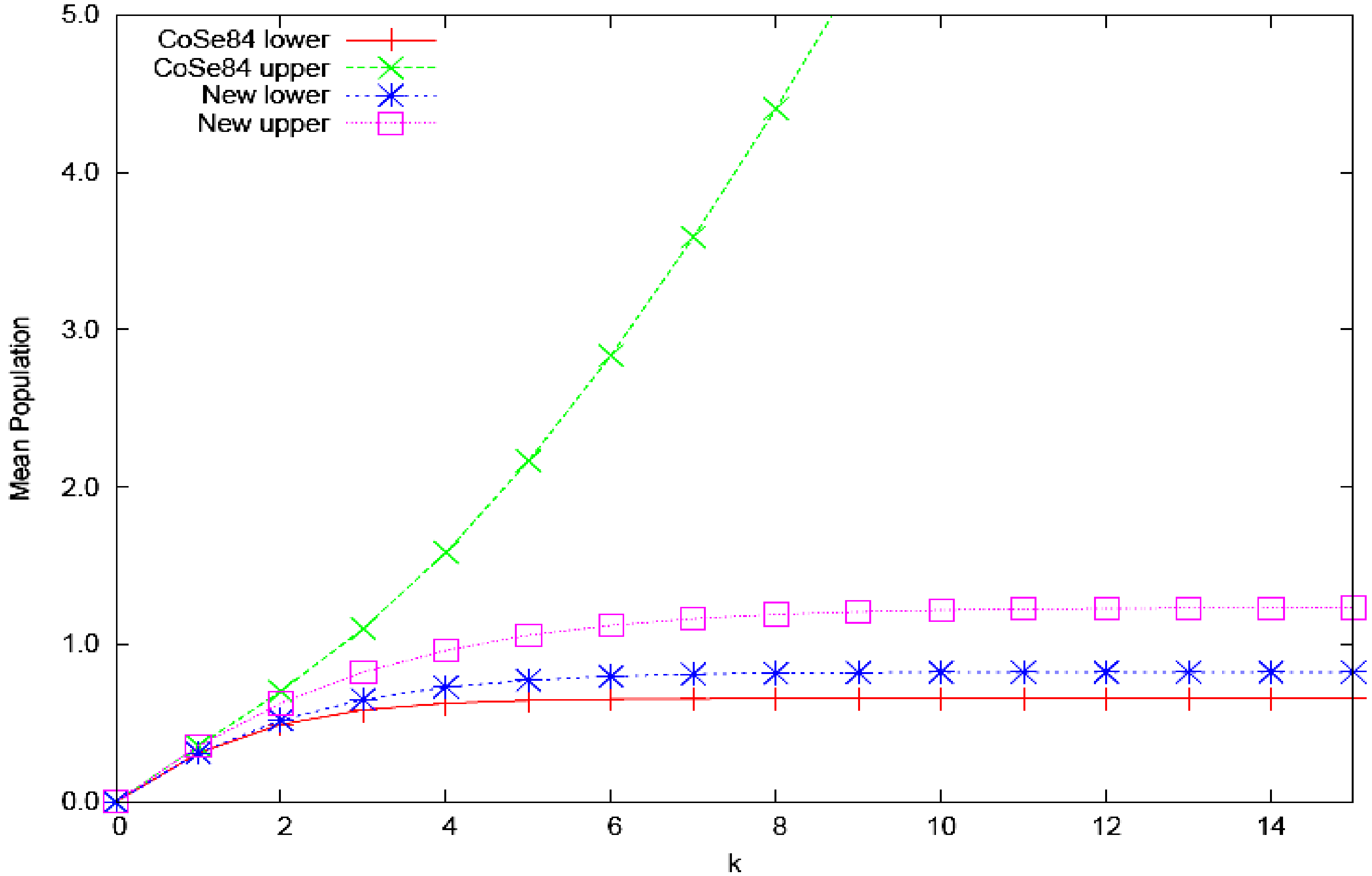
$$\begin{pmatrix} -\Sigma & \lambda^- & & & \\ \mu^- + \varepsilon_2 & -\Sigma & \lambda^- & & \\ & \ddots & \ddots & \ddots & \\ & & \mu^- + \varepsilon_2 & -\Sigma & \lambda^- \\ & & & \mu^- + \varepsilon_2 & -\Sigma \end{pmatrix}$$

Matrix yielding the upper bound for the rewards (diagonal elements are not printed)

$$\begin{pmatrix} -\Sigma & \lambda^- + \varepsilon_1 & & & \\ \mu^- & -\Sigma & \lambda^- + \varepsilon_1 & & \\ & \ddots & \ddots & \ddots & \\ & & \mu^- & -\Sigma & \lambda^- + \varepsilon_1 \\ & & & \mu^- & -\Sigma \end{pmatrix}$$

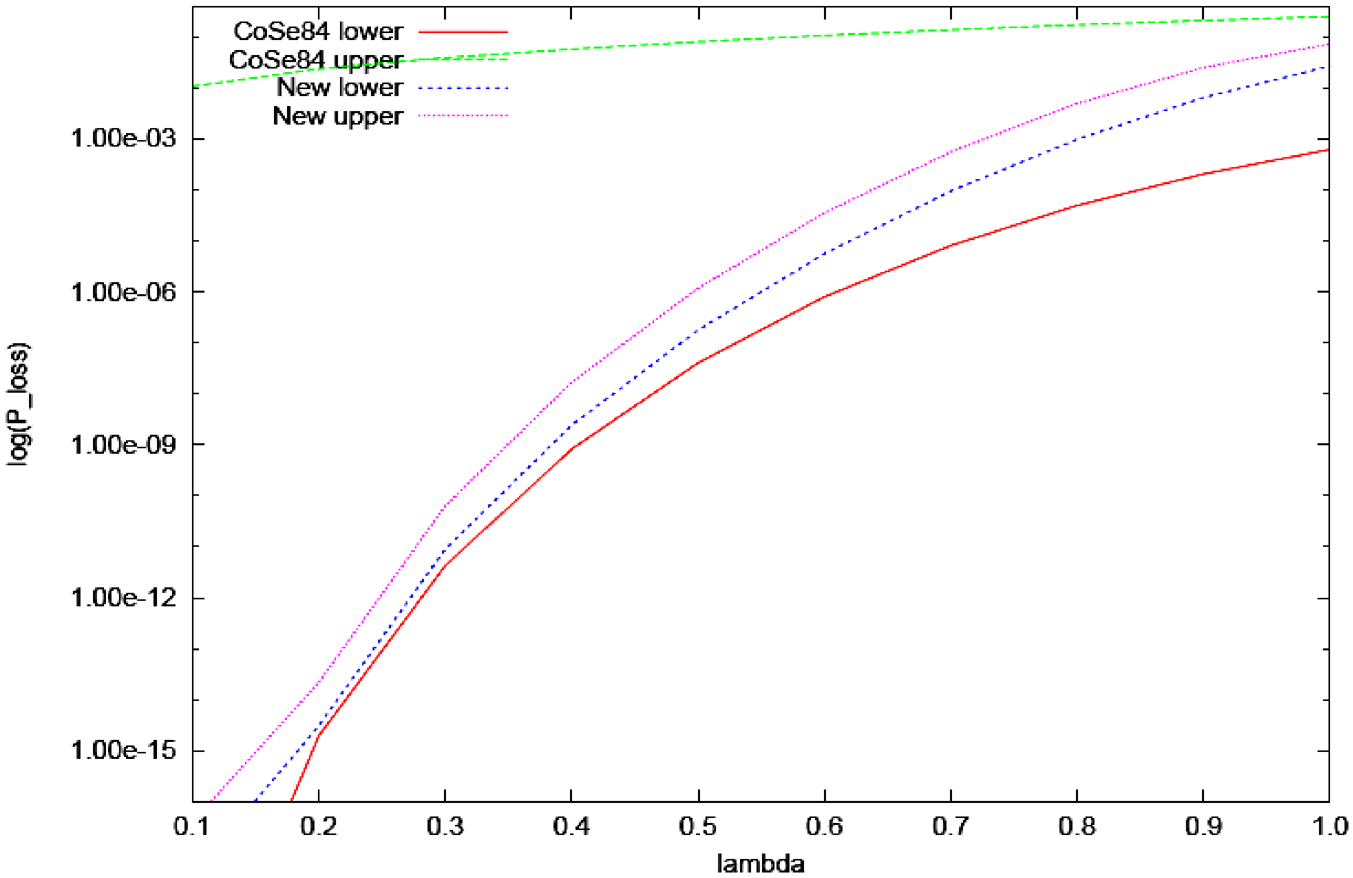
Mean population for varying K

($\lambda^- = 0.475$, $\varepsilon_1 = 0.05$, $\mu^- = 0.95$, $\varepsilon_2 = 0.1$)



Improved Bounds

Blocking probability for varying λ^- ($K = 20$, $\varepsilon_1 = 0.1 \cdot \lambda^-$, $\mu^- = 0.95$, $\varepsilon_2 = 0.1$)



Improved Bounds

If $\mathbf{L} \neq \mathbf{U}$, then $\mathbb{P}_{\mathbf{L,U}}$ is an infinite set!

- For the computation of bounds we need to generate two matrices
  - $\mathbf{P}^{-} \in \mathbb{P}_{\mathbf{L,U}}$ with steady state vector $\mathbf{v}^{-}$ such that
    $\mathbf{v}^{-}\mathbf{r} = \min_{\mathbf{v} \in \mathbb{V}_{\mathbf{L,U}}} (\mathbf{vr})$
  - $\mathbf{P}^{+} \in \mathbb{P}_{\mathbf{L,U}}$ with steady state vector $\mathbf{v}^{+}$ such that
    $\mathbf{v}^{+}\mathbf{r} = \max_{\mathbf{v} \in \mathbb{V}_{\mathbf{L,U}}} (\mathbf{vr})$

How to compute these matrices from an infinite set?

Improved Bounds

A matrix $\mathbf{P} \in \mathbb{P}_{\mathbf{L,U}}$ is an extremal point of $\mathbb{P}_{\mathbf{L,U}}$ iff

no two matrices $\mathbf{P}_1, \mathbf{P}_2 \in \mathbb{P}_{\mathbf{L,U}}$ ($\mathbf{P}_1 \neq \mathbf{P}_2$) exist

such that $\mathbf{P} = \beta\mathbf{P}_1 + (1-\beta)\mathbf{P}_2$ for $0 < \beta < 1$

$\mathbb{E}_{\mathbf{L,U}} = \{\mathbf{P} \mid \mathbf{P} \in \mathbb{P}_{\mathbf{L,U}}$ and $\mathbf{P}$ is extremal point of $\mathbb{P}_{\mathbf{L,U}}\}$

Theorem 1:

1. $\mathbf{P} \in \mathbb{E}_{\mathbf{L,U}} \Leftrightarrow \forall r \in \{1,\ldots,n\}\ \exists c_r \in \{1,\ldots,n\}$ such that
   $\mathbf{P}(r,i) = \mathbf{L}(r,i)$ or $\mathbf{P}(r,i) = \mathbf{U}(r,i)$ for all $i \neq c_r$
2. The set $\mathbb{E}_{\mathbf{L,U}}$ is finite and contains at most $(n!)^n$ matrices

Improved Bounds

Theorem 2:

1. $\exists \mathbf{P}^- \in \mathbb{E}_{\mathbf{L,U}}$ with steady state vector $\mathbf{v}^-$ such that
   $$\mathbf{v}^-\mathbf{r} = \min_{\mathbf{v}\in\mathbb{V}_{\mathbf{L,U}}} (\mathbf{vr})$$
2. $\exists \mathbf{P}^+ \in \mathbb{E}_{\mathbf{L,U}}$ with steady state vector $\mathbf{v}^+$ such that
   $$\mathbf{v}^+\mathbf{r} = \max_{\mathbf{v}\in\mathbb{V}_{\mathbf{L,U}}} (\mathbf{vr})$$

Where are we?

- The set can be enumerated and the maximum/minimum can be computed
- Unfortunately, the effort is too high even for small matrices

A new Bounding Algorithm

- Let Π be the set of permutations on {1,…,n}
- For $\pi \in \Pi$ generate a matrix **P** as follows in at most n steps:
  - Start with $\mathbf{P}^{(0)} = \mathbf{L}$
  - Let for m=1,…,n, i=1,…n and $j=\pi(m)$:

    $\mathbf{P}^{(m)}(i,j) = \min(\mathbf{U}(i,j), 1 - \Sigma_{k \neq i} \mathbf{P}^{(m-1)}(i,k))$
- Let $\mathbb{F}_{\mathbf{L,U}}$ be the set of all matrices P which are generated according to some $\pi \in \Pi$

Theorem 3:

1. $\mathbb{F}_{\mathbf{L,U}} \subseteq \mathbb{E}_{\mathbf{L,U}}$
2. $\mathbb{F}_{\mathbf{L,U}}$ contains at most n! matrices
3. $\mathbf{P}^-, \mathbf{P}^+ \in \mathbb{F}_{\mathbf{L,U}}$

A new Bounding Algorithm

For $\pi \in \Pi$ define

$\mathbb{H}^{(1)}(\pi) = \{\theta \mid \exists i,j: \pi(i)=\theta(j), \pi(j)=\theta(i) \text{ and } \forall k \neq i,j: \pi(k)=\theta(k)\}$

- ➢ $\mathbb{H}^{(1)}(\pi)$ contains at most n(n-1)/2 permutations

Theorem 4:

Let $\mathbf{P}_\pi \in \mathbb{F}_{\mathbf{L,U}}$ be generated using permutation π and let $\mathbf{v}_\pi$ the corresponding stationary vector, then

1. if $\mathbf{v}_\pi\mathbf{r} \leq \mathbf{v}_\theta\mathbf{r}$ for all $\theta \in \mathbb{H}^{(1)}(\pi) \Rightarrow \mathbf{v}_\pi\mathbf{r} = \min_{\mathbf{v}\in\mathbb{V}_{\mathbf{L,U}}} (\mathbf{vr})$

2. if $\mathbf{v}_\pi\mathbf{r} \geq \mathbf{v}_\theta\mathbf{r}$ for all $\theta \in \mathbb{H}^{(1)}(\pi) \Rightarrow \mathbf{v}_\pi\mathbf{r} = \max_{\mathbf{v}\in\mathbb{V}_{\mathbf{L,U}}} (\mathbf{vr})$

A new Bounding Algorithm

<u>Outline of an algorithm</u> <u>(here computation of the upper bound):</u>

1. Choose one $\pi \in \Pi$ and compute $\mathbf{P}_\pi$
2. While $\theta \in \mathbb{H}^{(1)}(\pi)$ exist such that $\mathbf{v}_\pi \mathbf{r} \leq \mathbf{v}_\theta \mathbf{r}$ do
   i. $\pi = \theta$
3. $\mathbf{v}_\pi \mathbf{r}$ is the upper bound for the reward

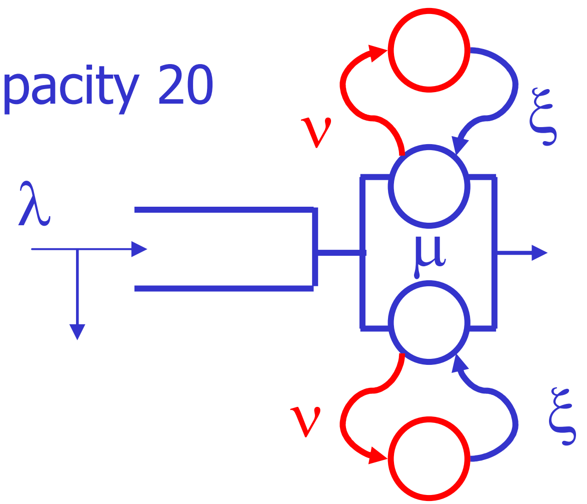
Effort of the approach

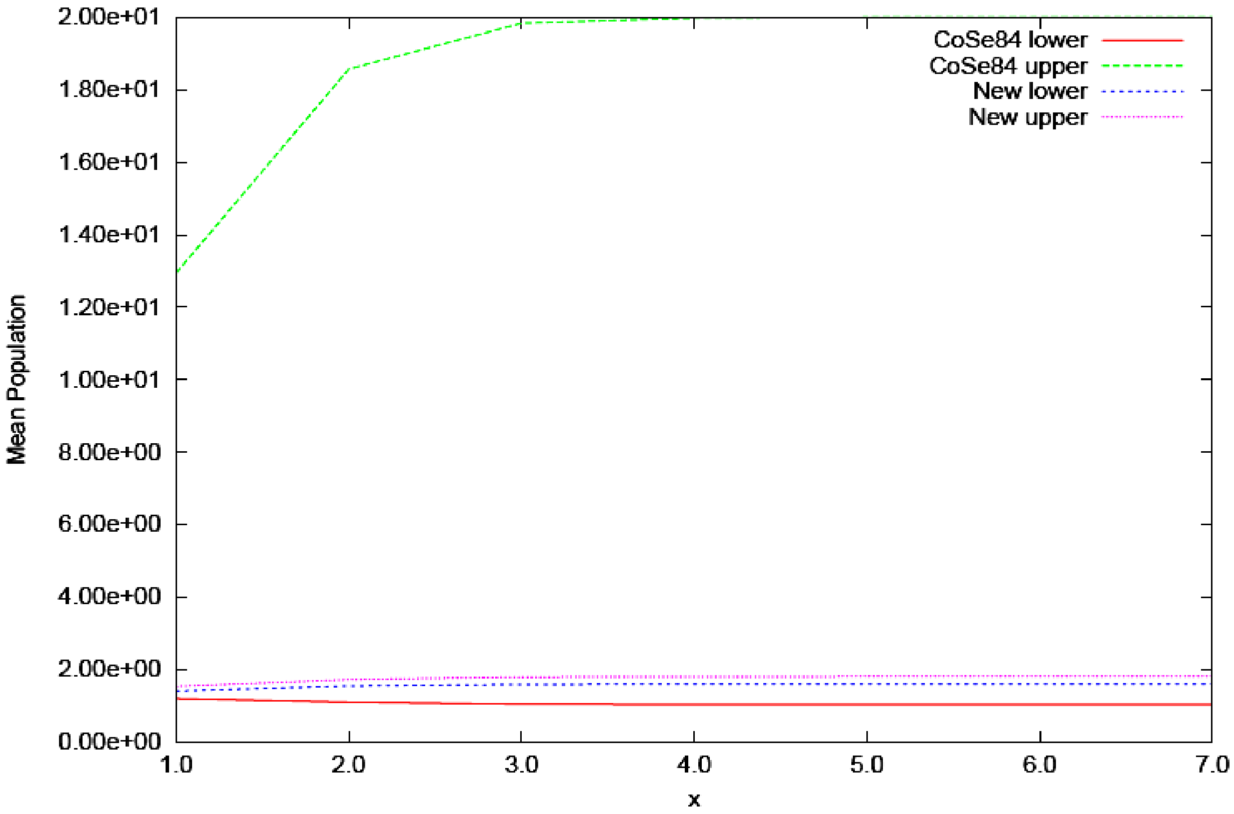
- Analysis of a new permutation in step 2 requires rank one update of an inverse matrix for all columns (worst case effort $O(n^3)$)
- Number of permutations π that is considered can be in $O(n!)$

Worst case complexity is horrible,
but average case seems to be much better!

A new Bounding Algorithm

Capacity 20





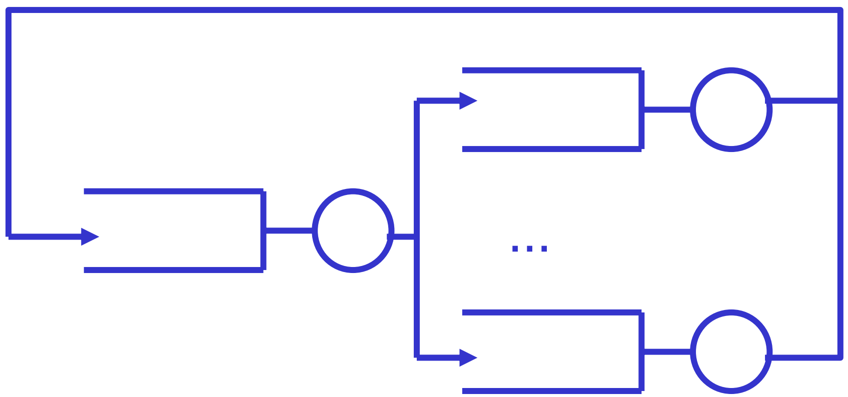
$\lambda \in [0.792, 0.808]$

$\mu \in [0.990, 1.010]$

$\xi \in [0.99 \cdot 10^{-x}, 1.01 \cdot 10^{-x}]$

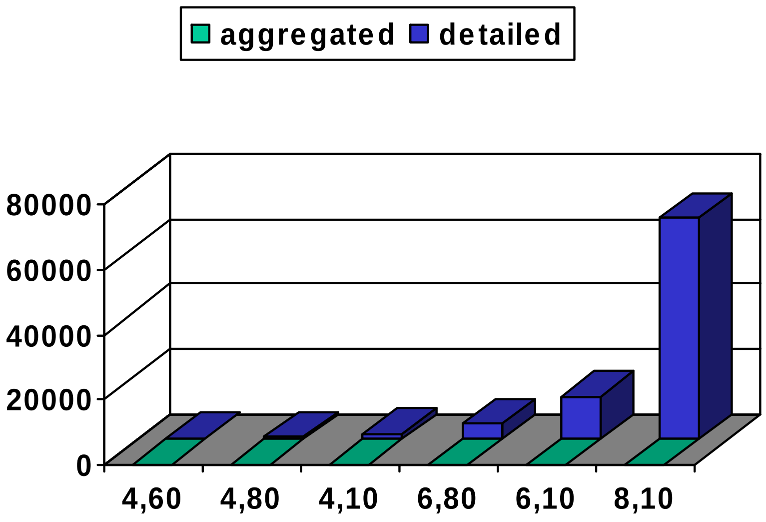
$\nu \in [0.99 \cdot 10^{-x-1}, 1.01 \cdot 10^{-x-1}]$

A new Bounding Algorithm

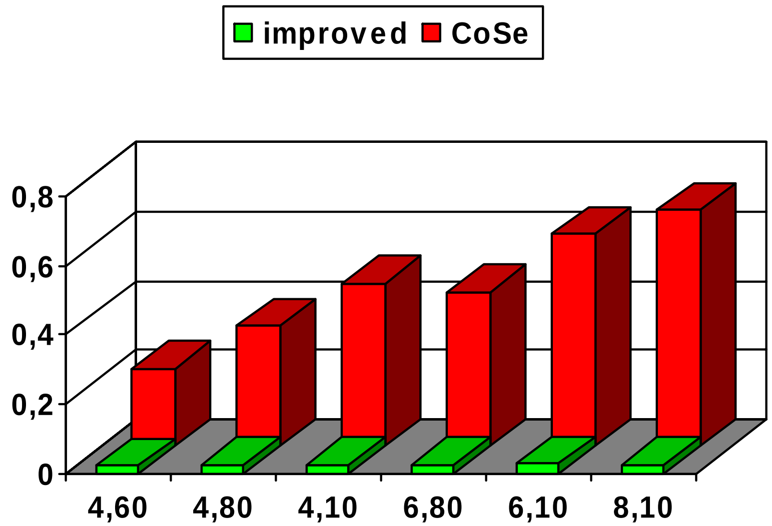


- Erlang 2 distribution at the central station
- K nearly lumpable peripheral stations
- N customers

State space detailed/aggregated (K,N)



Spread of the bounds for the throughput



Conclusions and Outlook

- ✓ Bounding matrices in a polyhedron of matrices
- ✓ Algorithm to compute bounding matrices
- ✓ Bounds are sharp (and improve known bounds significantly)
- ✓ Effort for bound computation is high

- ➢ Further Investigation of the algorithm
- ➢ Application of the approach for decomposition and aggregation
- ➢ More efficient methods for specific classes of matrices