Time-Based Maintenance Models under Uncertainty

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Abstract. Model based computation of optimal maintenance strategies is one of the classical applications of Markov Decision Processes. Unfortunately, a Markov Decision Process often does not capture the behavior of a component or system of components correctly because the duration of different operational phases is not exponentially distributed and the status of component is often only partially observable during operational times. The paper presents a general model for components with partially observable states and non-exponential failure, maintenance and repair times which are modeled by phase type distributions. Optimal maintenance strategies are computed using Markov decision theory. However, since the internal state of a component is not completely known, only bounds for the parameters of a Markov decision process can be computed resulting in a bounded parameters Markov decision process. For this kind of process optimal strategies can be computed assuming best, worst or average case behavior.

Keywords: Maintenance Models, Markov Decision Processes, Stochastic Dynamic Programming, Numerical Methods

1 Appendix: Proofs

Proof of Theorem 1 from [1] Let $\boldsymbol{\eta} = \boldsymbol{\nu}_i e^{t^* \boldsymbol{D}_i} / \left(\boldsymbol{\nu}_i e^{t^* \boldsymbol{D}_i} \mathbb{1} \right)$ for some $t^* \geq 0$ the distribution among the states of phase i at time $l \cdot \Delta$. The behavior in the following interval can be described exactly by the matrix

$$\boldsymbol{B}_i = \begin{pmatrix} \boldsymbol{D}_i & \boldsymbol{d}_i \boldsymbol{b}_i \\ \boldsymbol{0} & \boldsymbol{Q}_0 \end{pmatrix}$$

and $\boldsymbol{\xi} = (\boldsymbol{\eta}, \boldsymbol{0})\, e^{\Delta \boldsymbol{B}_i}$ is then the distribution at time $(l+1) \cdot \Delta$.

$\boldsymbol{\xi} = \left(\boldsymbol{\xi}_{-1}, \boldsymbol{\xi}_0, \ldots, \boldsymbol{\xi}_{2N} \right)$, where $\boldsymbol{\xi}_{-1}$ contains the probabilities of the state of phase i under the condition that the process remains in i during the whole interval and $\boldsymbol{\xi}_j$ $(0 \leq j \leq 2N)$ contains the probabilities of states in phase j which means that phase i has been left.

We present the proof for the lower bound, the proof for the upper bound is completely analogous. We have

$$e^{\Delta \boldsymbol{B}_i} = \begin{pmatrix} e^{\Delta \boldsymbol{D}_i} & e^{\Delta \boldsymbol{D}_i} \boldsymbol{d}_i \boldsymbol{b}_i \\ \boldsymbol{0} & e^{\Delta \boldsymbol{Q}_0} \end{pmatrix} \quad \text{and}$$
$$e^{\Delta \boldsymbol{B}_i^-} = \begin{pmatrix} e^{-\Delta \lambda_i^+} & e^{-\Delta \lambda_i^+} \lambda_i^- \boldsymbol{b}_i \\ \boldsymbol{0} & e^{\Delta \boldsymbol{Q}_0} \end{pmatrix}.$$

By assumption $\boldsymbol{\xi} = (\boldsymbol{\eta}, \boldsymbol{0})\, e^{\Delta \boldsymbol{B}_i}$ and $\boldsymbol{\phi}^- = (1, \boldsymbol{0})\, e^{\Delta \boldsymbol{B}_i^-}$. Furthermore, let $\lambda_i(t) = \boldsymbol{\nu} e^{t\boldsymbol{D}_i} \boldsymbol{d}_i / (\boldsymbol{\nu} e^{t\boldsymbol{D}_i} \mathbb{1})$. Then $\boldsymbol{\nu} e^{t\boldsymbol{D}_i} \mathbb{1} = 1 - \int_0^t e^{-\tau \lambda_i(t)} \lambda_i(t) d\tau \geq 1 - e^{-t\lambda_i^+}$ which implies that $\boldsymbol{\xi}_{-1} \mathbb{1} \geq \boldsymbol{\phi}_{-1}^-$.

For the remaining elements of the vector observe that lower right submatrices of $\boldsymbol{B}_i$ and $\boldsymbol{B}_i^-$ equal $\boldsymbol{Q}_0$ and that all elements in $e^{\Delta \boldsymbol{Q}_0}$ are non-negative. Thus, the sub-vector $(\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_{2N})$ and $(\boldsymbol{\phi}_1, \ldots, \boldsymbol{\phi}_{2N})$ can both be computed from

$$\int_0^{\Delta} \mu_\tau \boldsymbol{b}_i e^{(\Delta - \tau)\boldsymbol{Q}_0} d\tau$$

where μ_τ is the flow from the first block (first visit of phase i) into the remaining states. For matrix $\boldsymbol{B}$ μ_τ equals

$$\mu_\tau = \boldsymbol{\eta} e^{\tau \boldsymbol{D}_i} \boldsymbol{d}_i \geq \boldsymbol{\eta} e^{\tau \boldsymbol{D}_i} \mathbb{1} \lambda_i^- \geq (1 - e^{-\tau \lambda_i^+}) \lambda_i^- = \mu_\tau^-.$$

μ_τ^- the flow into the second block for $\boldsymbol{B}_i^-$. The vectors at time $(l+1)\Delta$ are then given by

$$\boldsymbol{\phi}_{0:2N} = \int_0^{\Delta} \mu_\tau \boldsymbol{b}_i e^{(\Delta - \tau)\boldsymbol{Q}_0} \quad \text{and}$$
$$\boldsymbol{\phi}_{0:2N}^- = \int_0^{\Delta} \mu_\tau^- \boldsymbol{b}_i e^{(\Delta - \tau)\boldsymbol{Q}_0}$$

Since all entries of the matrices and vectors are non-negative, a smaller flow into the submatrix results in elementwise smaller values in the integral above.

Proof of Theorem 2 **from** [1] The proof is similar to the proof for Theorem 1 presented above. Again we show the proof for the lower bound. The reward for state j in phase i is given by $r_j + p_i \boldsymbol{d}_i(j) + r_{pos}$. Consequently, r_i^- is a lower bound for this reward. It has already been shown in the proof for Theorem 2 that the probability of being in phase i without leaving it is smaller in the process described by $\boldsymbol{B}_i^-$, then in the process described by $\boldsymbol{B}_i$. This implies that also the accumulated reward is smaller.

The lower right sub-matrix is $\boldsymbol{Q}_0$ in both matrices $\boldsymbol{B}_i^-$ and $\boldsymbol{B}_i$ and the reward vector is in both cases $\boldsymbol{r}$. Let

$$\acute{\boldsymbol{\phi}}_\tau = \int_0^\tau \boldsymbol{\eta} e^{-x\boldsymbol{D}_i} \boldsymbol{d}_i \boldsymbol{b}_i e^{-(\tau - x)\boldsymbol{Q}_0} dx,$$
$$\acute{\boldsymbol{\phi}}_\tau^- = \int_0^\tau e^{-x\lambda_i^+} \lambda_i^- \boldsymbol{b}_i e^{-(\tau - x)\boldsymbol{Q}_0} dx.$$

It follows from the proof of Theorem 2 that $\acute{\boldsymbol{\phi}}_\tau^- \leq \acute{\boldsymbol{\phi}}_\tau$ which implies that also $\acute{\boldsymbol{\phi}}_\tau^- \boldsymbol{r} \leq \acute{\boldsymbol{\phi}}_\tau \boldsymbol{r}$ which equal the accumulated rewards.

References

1. P.Buchholz, I.Dohndorf, and D. Scheftelowitsch. Time-based maintenance models under uncertainty. 2017.