Time-Based Maintenance Models under Uncertainty

Peter Buchholz, Iryna Dohndorf, Dimitri Scheftelowitsch

TU Dortmund, Germany

Abstract. Model based computation of optimal maintenance strategies is one of the classical applications of Markov Decision Processes. Unfortunately, a Markov Decision Process often does not capture the behavior of a component or system of components correctly because the duration of different operational phases is not exponentially distributed and the status of component is often only partially observable during operational times. The paper presents a general model for components with partially observable states and non-exponential failure, maintenance and repair times which are modeled by phase type distributions. Optimal maintenance strategies are computed using Markov decision theory. However, since the internal state of a component is not completely known, only bounds for the parameters of a Markov decision process can be computed resulting in a bounded parameters Markov decision process. For this kind of process optimal strategies can be computed assuming best, worst or average case behavior.

Keywords: Maintenance Models, Markov Decision Processes, Stochastic Dynamic Programming, Numerical Methods

1 Appendix: Proofs

Proof of Theorem 1 **from [1]** Let $\boldsymbol{\eta} = \boldsymbol{\nu}_i e^{t^* \boldsymbol{D}_i} / \left(\boldsymbol{\nu}_i e^{t^* \boldsymbol{D}_i} \mathbb{1} \right)$ for some $t^* \geq 0$ the distribution among the states of phase i at time $l \cdot \Delta$. The behavior in the following interval can be described exactly by matrix

$$\boldsymbol{B}_i = \begin{pmatrix} \boldsymbol{D}_i & \boldsymbol{d}_i \boldsymbol{b}_i \\ \boldsymbol{0} & \boldsymbol{Q} \end{pmatrix}$$

and $\boldsymbol{\xi} = (\boldsymbol{\eta}, \boldsymbol{0})\, e^{\Delta \boldsymbol{B}_i}$ is then the distribution at time $(l+1) \cdot \Delta$.

$\boldsymbol{\xi} = \left(\boldsymbol{\xi}_{-1}, \boldsymbol{\xi}_0, \ldots, \boldsymbol{\xi}_{2N} \right)$, where $\boldsymbol{\xi}_{-1}$ contains the probabilities of the state of phase i under the condition that the process remains in i during the whole interval and $\boldsymbol{\xi}_j$ $(0 \leq j \leq 2N)$ contains the probabilities of states in phase j which means that phase i has been left.

We present the proof for the lower bound, the proof for the upper bound is completely analogous. We have

$$e^{\Delta \boldsymbol{B}_i} = \begin{pmatrix} e^{\Delta \boldsymbol{D}_i} & e^{\Delta \boldsymbol{D}_i} \boldsymbol{d}_i \boldsymbol{b}_i \\ \boldsymbol{0} & e^{\Delta \boldsymbol{Q}} \end{pmatrix} \quad \text{and}$$
$$e^{\Delta \boldsymbol{B}_i^-} = \begin{pmatrix} e^{-\Delta \lambda_i^+} & e^{-\Delta \lambda_i^+} \lambda_i^- \boldsymbol{b}_i \\ \boldsymbol{0} & e^{\Delta \boldsymbol{Q}} \end{pmatrix}.$$

By assumption $\boldsymbol{\xi} = (\boldsymbol{\eta}, \boldsymbol{0})\, e^{\Delta \boldsymbol{B}_i}$ and $\boldsymbol{\phi}^- = (1, \boldsymbol{0})\, e^{\Delta \boldsymbol{B}_i^-}$. Furthermore, let $\lambda_i(t) = \boldsymbol{\nu} e^{t\boldsymbol{D}_i} \boldsymbol{d}_i / (\boldsymbol{\nu} e^{t\boldsymbol{D}_i} \mathbb{1})$. Then $\boldsymbol{\nu} e^{t\boldsymbol{D}_i} \mathbb{1} = 1 - \int_0^t e^{-\tau \lambda_i(t)} \lambda_i(t) d\tau \geq 1 - e^{-t\lambda_i^+}$ which implies that $\boldsymbol{\xi}_{-1} \mathbb{1} \geq \boldsymbol{\phi}_{-1}^-$.

For the remaining elements of the vector observe that lower right submatrices of $\boldsymbol{B}_i$ and $\boldsymbol{B}_i^-$ equal $\boldsymbol{Q}$ and that all elements in $e^{\Delta \boldsymbol{Q}}$ are non-negative. Thus, the sub-vector $(\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_{2N})$ and $(\boldsymbol{\phi}_1, \ldots, \boldsymbol{\phi}_{2N})$ can both be computed from

$$\int_0^{\Delta} \mu_\tau \boldsymbol{b}_i e^{(\Delta - \tau)\boldsymbol{Q}} d\tau$$

where μ_τ is the flow from the first block (first visit of phase i) into the remaining states. For matrix $\boldsymbol{B}$ μ_τ equals

$$\mu_\tau = \boldsymbol{\eta} e^{\tau \boldsymbol{D}_i} \boldsymbol{d}_i \geq \boldsymbol{\eta} e^{\tau \boldsymbol{D}_i} \mathbb{1} \lambda_i^- \geq (1 - e^{-\tau \lambda_i^+}) \lambda_i^- = \mu_\tau^-.$$

μ_τ^- the flow into the second block for $\boldsymbol{B}_i^-$. The vectors at time $(l+1)\Delta$ are then given by

$$\boldsymbol{\phi}_{0:2N} = \int_0^{\Delta} \mu_\tau \boldsymbol{b}_i e^{(\Delta - \tau)\boldsymbol{Q}} \quad \text{and}$$
$$\boldsymbol{\phi}_{0:2N}^- = \int_0^{\Delta} \mu_\tau^- \boldsymbol{b}_i e^{(\Delta - \tau)\boldsymbol{Q}}$$

Since all entries of the matrices and vectors are non-negative, a smaller flow into the submatrix results in elementwise smaller values in the integral above.

Proof of Theorem 2 **from [1]** The proof is similar to the proof for Theorem 1 presented above. Again we show the proof for the lower bound.

The behavior of the component without knowledge of the detailed state in phase i is described by matrix $\hat{\boldsymbol{C}}_i$ from Eq. (9). If the initial distribution is known, then matrix

$$\boldsymbol{C} = \begin{pmatrix} \boldsymbol{D}_i & \boldsymbol{d}_i \boldsymbol{b}_i \\ \boldsymbol{0} & \boldsymbol{Q} \end{pmatrix}$$

and initial distribution $(\boldsymbol{\eta}, \boldsymbol{0})$, where η is the distribution over the phases of phase i at time $l \cdot \Delta$, can be used to compute the reward accumulated in $[l \cdot \Delta, (l+1) \cdot \Delta]$. Let $\boldsymbol{r}_i$ be a column vector of length m_i with value $r_i + d_i(j) r_0 + r_{pos}$ in position j, then

$$(\boldsymbol{\eta}, \boldsymbol{0}) \int_0^{\Delta} e^{\tau \boldsymbol{C}} d\tau \begin{pmatrix} \boldsymbol{r}_i \\ \boldsymbol{r} \end{pmatrix} \tag{1}$$

equals the reward accumulated in the interval. Let $\boldsymbol{\psi}^\tau = (\boldsymbol{\psi}_1^\tau, \boldsymbol{\psi}_2^\tau) = (\boldsymbol{\eta}, \mathbf{0})\, e^{\tau \boldsymbol{C}_i}$ be the probability at time τ, where $\boldsymbol{\psi}_1^\tau$ is of length m_i. Then the accumulated reward can be rewritten as

$$\int_0^\Delta \boldsymbol{\psi}_1^\tau \boldsymbol{r}_i d\tau + \int_0^\Delta \boldsymbol{\psi}_2^\tau \boldsymbol{r} d\tau.$$

Now let $\hat{\boldsymbol{\psi}}^\tau = \left(\hat{\psi}_1^\tau, \hat{\boldsymbol{\psi}}_2^\tau, \hat{\psi}_3^\tau\right) = (1, \mathbf{0})\, e^{\tau \hat{\boldsymbol{C}}_i}$ be the probability at time τ for the process with matrix $\hat{\boldsymbol{C}}_i$, where $\hat{\psi}_1^\tau$ and $\hat{\psi}_3^\tau$ are scalars. The accumulated reward for this process is given by

$$\int_0^\Delta \hat{\psi}_1^\tau r_i^- d\tau + \int_0^\Delta \hat{\boldsymbol{\psi}}_2^\tau \boldsymbol{r} d\tau + \int_0^\Delta \hat{\psi}_1^\tau u_i^- d\tau.$$

Since $\boldsymbol{C}_i$ and $\hat{\boldsymbol{C}}_I$ are generator matrices, the vectors $\boldsymbol{\psi}^\tau$ and $\hat{\boldsymbol{\psi}}^\tau$ are both probability vectors. Using the same proof as for Theorem 1, it can be shown that $\boldsymbol{\psi}_1^\tau \mathbb{1} \geq \hat{\psi}_1^\tau$ and $\boldsymbol{\psi}_2^\tau \geq \hat{\boldsymbol{\psi}}_2^\tau$. Furthermore, $r_i^- \leq \min_j \boldsymbol{r}_i(j)$ and $u^- \leq \min_j \boldsymbol{r}(j)$. Then

$$\begin{array}{lr}
\int_0^\Delta \boldsymbol{\psi}_1^\tau \boldsymbol{r}_i d\tau + \int_0^\Delta \boldsymbol{\psi}_2^\tau \boldsymbol{r} d\tau & \geq \\
\int_0^\Delta \hat{\psi}_1^\tau r_i^- d\tau + \int_0^\Delta (\boldsymbol{\psi}_1^\tau \mathbb{1} - \hat{\psi}_1^\tau) r_i^- d\tau + \int_0^\Delta \hat{\boldsymbol{\psi}}_2^\tau \boldsymbol{r} d\tau + \int_0^\Delta (\boldsymbol{\psi}_2^\tau - \hat{\boldsymbol{\psi}}_2^\tau) \boldsymbol{r} d\tau & \geq \\
\int_0^\Delta \hat{\psi}_1^\tau r_i^- d\tau + \int_0^\Delta \hat{\boldsymbol{\psi}}_2^\tau \boldsymbol{r} d\tau + \int_0^\Delta \left(\boldsymbol{\psi}_1^\tau \mathbb{1} + \boldsymbol{\psi}_2^\tau \mathbb{1} - \hat{\psi}_1^\tau - \hat{\boldsymbol{\psi}}_2^\tau\right) u_i^- & = \\
\int_0^\Delta \hat{\psi}_1^\tau r_i^- d\tau + \int_0^\Delta \hat{\boldsymbol{\psi}}_2^\tau \boldsymbol{r} d\tau + \hat{\psi}_1^\tau u_i^- d\tau. &
\end{array}$$

The last equality follows from the fact that $\hat{\boldsymbol{\psi}}^\tau$ is a probability vector.

References

1. P.Buchholz, I.Dohndorf, and D. Scheftelowitsch. Time-based maintenance models under uncertainty. 2017.